Geometry of graphs of multicurves

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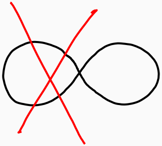
IHES

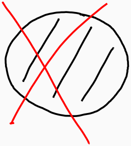
Basic objects

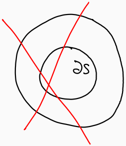
Surface: connected, compact, orientable

$S = S_{g,b}$ genus g, b boundary components

Curve: simple essential non-peripheral

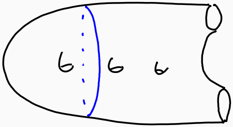


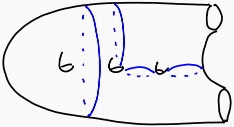




considered up to isotopy

Multicurve: collection of disjoint curves





Mapping class group: $\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/\text{isotopy}$

Graphs of multicurves

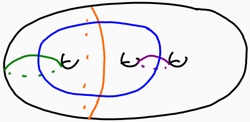
Graph of multicurves $\mathcal{G}(S)$:

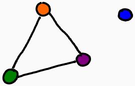
- vertices represent multicurves in S
- edges of length 1

Examples

Curve graph $\mathcal{C}(S)$:

- vertices: curves in S
- edges: disjointness





$\mathrm{MCG}(S) \curvearrowright \mathcal{C}(S)$

Masur–Minsky '99: $\mathcal{C}(S)$ is δ-hyperbolic and infinite diameter.

Graphs of multicurves

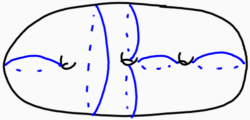
Graph of multicurves $\mathcal{G}(S)$:

- vertices represent multicurves in S
- edges of length 1

Examples

Pants graph $\mathcal{P}(S)$:

- vertices: pants decompositions of S



Brock '03: $\mathcal{P}(S) \sim_{\mathsf{QI}} (\mathrm{Teich}(S), d_{\mathrm{WP}})$.

Graphs of multicurves

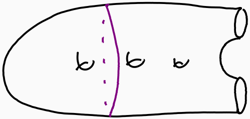
Graph of multicurves $\mathcal{G}(S)$:

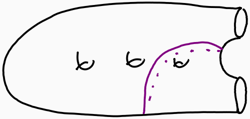
- vertices represent multicurves in S
- edges of length 1

Examples

Separating curve graph $\mathsf{Sep}(S)$:

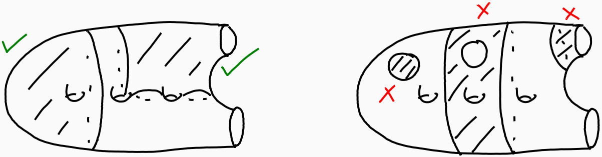
- vertices: separating curves in S
- edges: disjointness





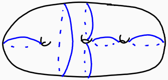
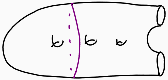
Subsurfaces

Essential:



A subsurface $W \subseteq S$ is a **witness** for $\mathcal{G}(S)$ if every vertex of $\mathcal{G}(S)$ intersects W (and $W \neq S_{0,3}$).

Examples:

	Graph	**Witnesses**
	$\mathcal{C}(S)$	$\{S\}$
	$\mathcal{P}(S)$	, and bigger
	$\mathrm{Sep}(S)$	

Hierarchical graphs of multicurves

We'll say a graph of multicurves $\mathcal{G}(S)$ is **hierarchical** if:

- $\mathcal{G}(S)$ is connected
- the action of $\mathsf{MCG}(S)$ on the set of curves in S induces an isometric action $\mathsf{MCG}(S) \curvearrowright \mathcal{G}(S)$
- $\mathcal{G}(S)/\mathsf{MCG}(S)$ is compact
- no witness for $\mathcal{G}(S)$ is an annulus.

Examples: curve graph, pants graph, separating curve graph

Theorem (V. '18)

Let $\mathcal{G}(S)$ be a hierarchical graph of multicurves. Then $\mathcal{G}(S)$ is a **hierarchically hyperbolic space**.

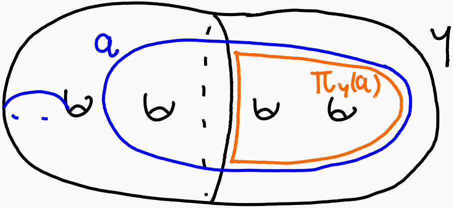
Lemma

Up to quasi-isometry, these graphs are determined by their set of witnesses.

Hierarchically hyperbolic spaces

Background:

Masur–Minsky '00: studied geometry of MCG(S) using **subsurface projections** to curve graphs of subsurfaces.

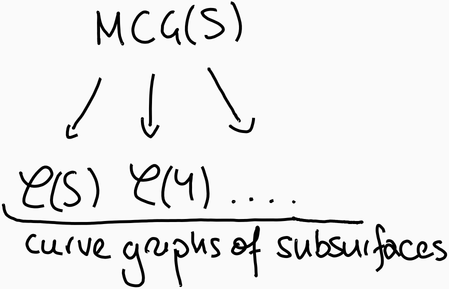


- project from a graph quasi-isometric to MCG(S)
- project to curve graphs of all (non-$S_{0,3}$) subsurfaces
- find distance estimate for MCG(S)

Hierarchically hyperbolic spaces

Hierarchically hyperbolic spaces (HHS):

- defined by Behrstock, Hagen and Sisto (2014)
- generalisation of Masur–Minsky's work on MCG(S)



- subsurface projections are Lipschitz
- various consistency properties

Hierarchically hyperbolic spaces

Hierarchically hyperbolic spaces (HHS):

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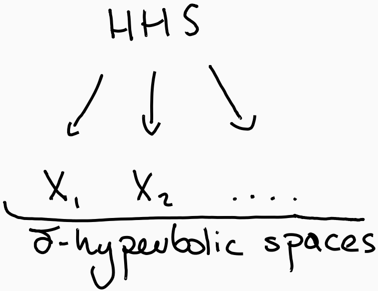


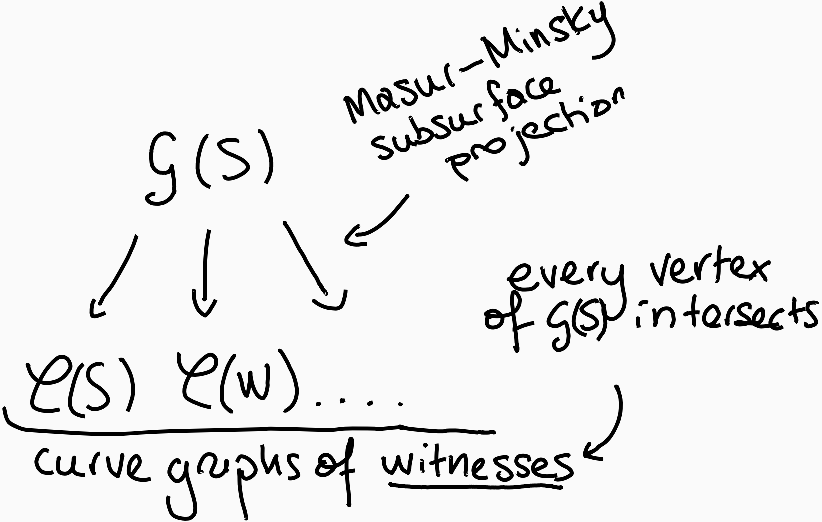
- relations generalising disjointness, nesting, overlapping
- coarsely Lipschitz projections satisfying a list of axioms
- a point in the HHS is coarsely determined by its projections

Hierarchically hyperbolic spaces

Hierarchically hyperbolic spaces (HHS):

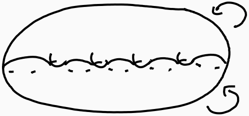
- defined by Behrstock, Hagen and Sisto (2014)
- generalisation of Masur–Minsky's work on $\mathrm{MCG}(S)$
- many results that hold for $\mathrm{MCG}(S)$ can then be generalised
  - distance estimate in terms of projection distances
  - structure of product regions

HHS structure on hierarchical graph of multicurves



Consequence: rank and hyperbolicity

We get quasiflats $\mathbb{Z}^n \hookrightarrow_{\mathsf{QI}} \mathcal{G}(S)$ from disjoint witnesses.



Idea:

- pseudo Anosov mapping classes act loxodromically on the curve graph (Masur–Minsky '99)
- when W is a witness, the projection $\mathcal{G}(S) \to \mathcal{C}(W)$ is Lipschitz
- conclude that partial pseudo Anosovs on witnesses act loxodromically on $\mathcal{G}(S)$
- for n disjoint witnesses get a quasi-isometrically embedded $\mathbb{Z}^n$

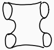
Rank

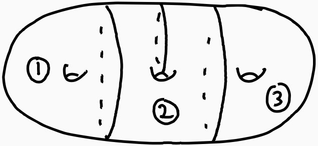
$$\text{rank}(\mathcal{G}(S)) = \max\{n \mid \mathbb{Z}^n \hookrightarrow_{\text{QI}} \mathcal{G}(S)\}$$

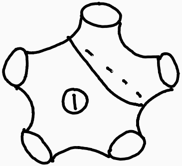
Corollary

The rank of $\mathcal{G}(S)$ is equal to the maximal cardinality of a set of pairwise disjoint witnesses for $\mathcal{G}(S)$.

Example (Behrstock–Minsky '08)

$\text{rank}(\mathcal{P}(S_{g,b})) = \text{max. } \#$ disjoint  and  $= \left\lfloor \frac{3g+b-3}{2} \right\rfloor$





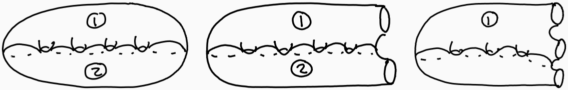
Rank

$$\mathrm{rank}(\mathcal{G}(S)) = \max\{n \mid \mathbb{Z}^n \hookrightarrow_{\mathrm{QI}} \mathcal{G}(S)\}$$

Corollary

The rank of $\mathcal{G}(S)$ is equal to the maximal cardinality of a set of pairwise disjoint witnesses for $\mathcal{G}(S)$.

Example $\mathrm{Sep}(S_{g,b})$



rank=2 if $b \leq 2$

rank=1 if $b \geq 3$

Hyperbolicity

Corollary

If $\text{rank}(\mathcal{G}(S)) = 1$ then $\mathcal{G}(S)$ is δ-hyperbolic.

Example: $\text{Sep}(S_{g,b})$ is δ-hyperbolic if $b \geq 3$.

General question: Given $\mathcal{G}(S)$, how does the geometry of the graph change as we vary S?

Example: Curve graph $\mathcal{C}(S)$ is always hyperbolic, $\text{Sep}(S)$ is not.

Classification

joint with Jacob Russell

Specialise to the separating curve graph.

Theorem

Let $S = S_{g,b}$ ($g \geq 3$).

- [V.] If $b \geq 3$, then $\mathrm{Sep}(S)$ is δ-hyperbolic.
- [Russell '19] If $b = 0, 2$, then $\mathrm{Sep}(S)$ is relatively hyperbolic.
- [Russell–V. '19] If $b = 1$, then $\mathrm{Sep}(S)$ is not relatively hyperbolic.

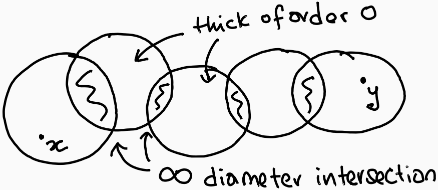
Thick metric spaces

Thick:

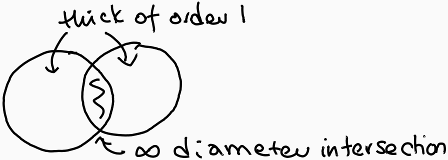
- defined by Behrstock–Druțu–Mosher
- obstruction to relative hyperbolicity
- inductive definition

Thick of order 0: Product of infinite diameter spaces.

Thick of order 1: $\forall x, y$



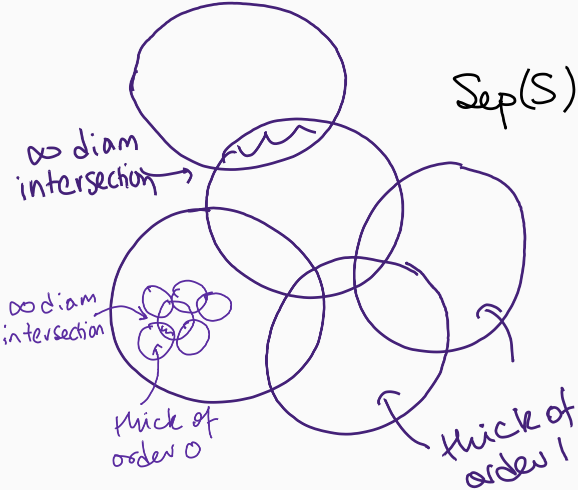
Thick of order 2:



...and so on

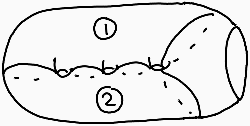
Thickness of Sep($S_{g,1}$)

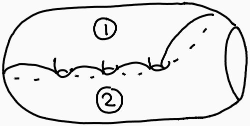
$S = S_{g,1}$: we show Sep(S) is thick of order at most 2.

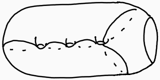
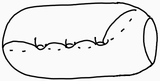


Product regions

Product regions (thick of order 0) come from pairs of disjoint witnesses.





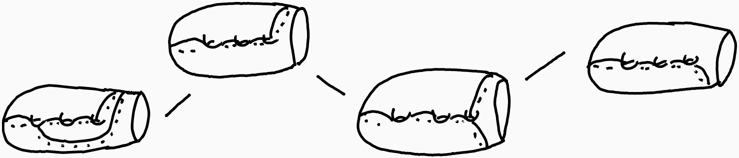
product region  $\subset$ product region 

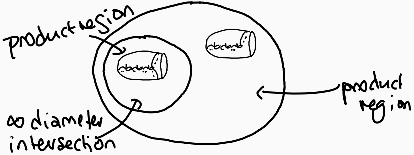
$\longrightarrow$ infinite diameter intersection

Thick of order 1 subsets

Define graph $\mathcal{DW}(S)$

- vertices: multicurves defining pairs of disjoint witnesses
- edges: inclusion





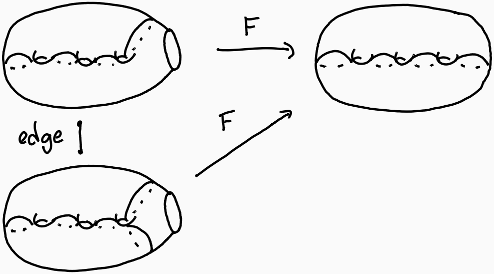
Edge $\longrightarrow$

Connected component $\longrightarrow$ thick of order 1 subset of $\mathrm{Sep}(S)$

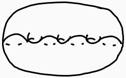
Thick of order 1 subsets

Want to understand the connected components of $\mathcal{DW}(S)$.

$F \colon S \to S_{g,0}$: cap boundary with a disc



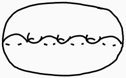
Lemma: The connected components of $\mathcal{DW}(S)$ are exactly the fibres of F.

thick of order 1 pieces $\longleftrightarrow$ MCG($S_{g,0}$)-translates of 

Thick of order ≤ 2

Want to chain together the thick of order 1 pieces.

Define a graph $\mathcal{F}(S_{g,0})$

- vertices: multicurves 
- edges: encode infinite diameter intersection of thick of order 1 pieces

Connected components of $\mathcal{F}(S_{g,0}) \longrightarrow$ thick of order ≤ 2 subsets of $\mathrm{Sep}(S)$

Lemma: $\mathcal{F}(S_{g,0})$ is connected.

$\longrightarrow \mathrm{Sep}(S)$ is thick of order ≤ 2.

Summary

- Hierarchical graphs of multicurves are hierarchically hyperbolic spaces.
- Consequence: a criterion for hyperbolicity and formula for rank by considering disjoint witnesses.
- With Russell, work on classifying (relative) hyperbolicity for these graphs in terms of HHS structure.

Thank you!