An Introduction to Mapping Class Groups

Katie Vokes

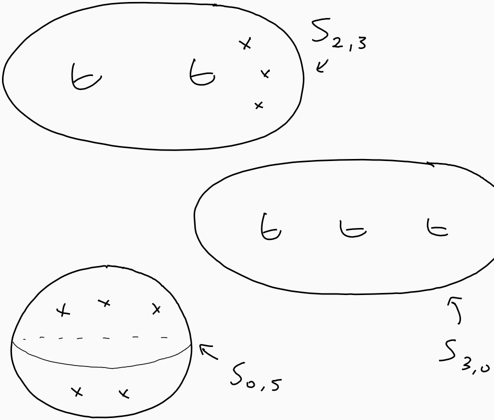
19 October 2020

IHES

Surface:

2-dimensional real manifold, connected, oriented and finite type

Classification of surfaces $\rightarrow S = S_{g,p}$: genus g surface with p points removed (p punctures), g and p finite (in $\mathbb{Z}_{\geq 0}$)



Definition of mapping class group

$\text{Homeo}^+(S) = \{\text{orientation-preserving homeomorphisms } S \to S\}$

$\text{Homeo}^+(S)$ forms a group under composition, but it is uncountable.

Mapping class group: $\text{MCG}(S) = \text{Homeo}^+(S)/\sim$

$f \sim g$ if f and g are **isotopic**. This means that there is a homotopy $F\colon S \times [0,1] \to S$ so that:

- $F(\cdot, 0) = f$
- $F(\cdot, 1) = g$
- $F(\cdot, t)$ is a homeomorphism for all t

Definition of mapping class group

Mapping class group: $\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/\sim$

$f \sim g$ if f and g are isotopic.

The mapping class group is a **countable** group, in fact it is **finitely presented**.

We will call an element of the mapping class group a **mapping class**. That is, a mapping class is an isotopy class of orientation-preserving self-homeomorphisms of S.

References

- Benson Farb, Dan Margalit, *A Primer on Mapping Class Groups*, Princeton Univ. Press, 2011
- Thomas Kwok-Keung Au, Feng Luo, Tian Yang, *Lectures on the mapping class group of a surface*, 2011, https://www.math.tamu.edu/~tianyang/lecture.pdf
- Yair Minsky, *A Brief Introduction to Mapping Class Groups*, PCMI lecture notes, 2011, https://gauss.math.yale.edu/~yhm3/research/PCMI.pdf
- Gwénaël Massuyeau, *A short introduction to mapping class groups*, 2009, https://massuyea.perso.math.cnrs.fr/notes/MCG.pdf

Surfaces with boundary

Sometimes we will allow surfaces to have boundary as well as/instead of punctures.

In this case, when defining $\mathsf{MCG}(S)$, we restrict to homeomorphisms that fix the boundary ∂S pointwise. The isotopies should also fix ∂S pointwise.

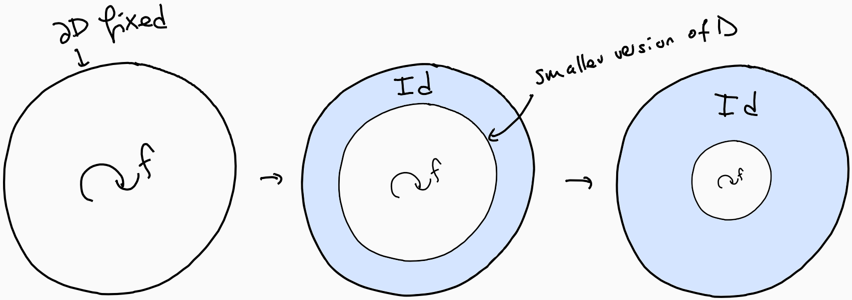
$$\mathsf{MCG}(S) = \mathsf{Homeo}^+(S, \partial S)/\sim$$

Example: MCG($\mathbf{D}$)

Let $\mathbf{D}$ be the closed disc.

$\mathrm{MCG}(\mathbf{D}) = 1$

That is, every homeomorphism $f\colon \mathbf{D} \to \mathbf{D}$ which fixes $\partial\mathbf{D}$ pointwise is isotopic to $\mathrm{Id}_{\mathbf{D}}$.



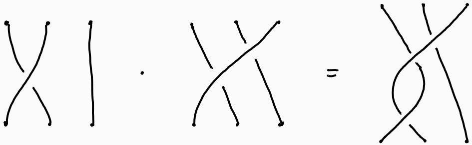
This is sometimes called the **Alexander trick**.

Example: $\mathrm{MCG}(\mathbf{D})$

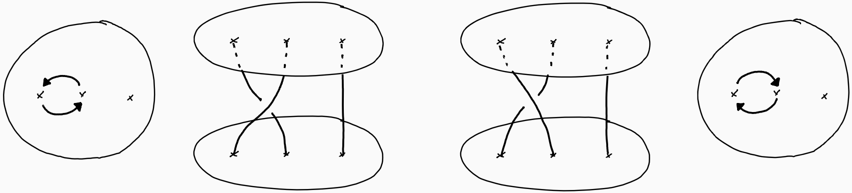
The fact that $\mathrm{MCG}(\mathbf{D})$ is trivial turns out to be very useful. We will see later an important tool that involves cutting a surface into topological discs and then applying the fact that discs have trivial mapping class group.

If we remove one point from $\mathbf{D}$ (add a puncture) the mapping class group is still trivial.

Example: braid groups

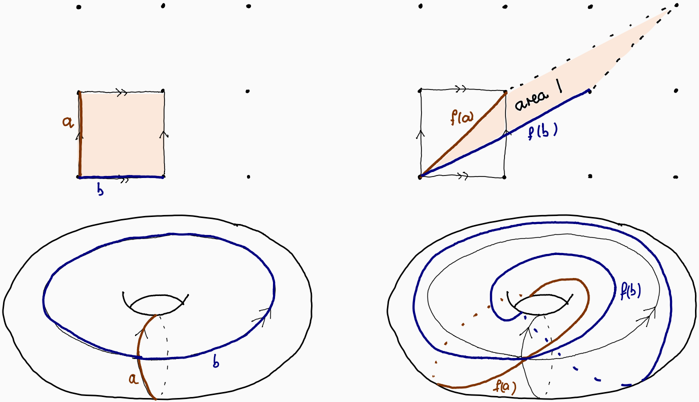


The braid group on n strands is equal to the mapping class group of the n times punctured disc.



Example: MCG(T^2)

We can think of T^2 as a quotient $\mathbb{R}^2/\sim$, where $(x, y) \sim (x + 1, y)$, $(x, y) \sim (x, y + 1)$.



It turns out that: MCG(T^2) $\cong$ SL$(2, \mathbb{Z})$.

Studying the mapping class group: curves

A **curve** in S is an embedding of the circle $a \colon S^1 \to S$.

A curve c is **separating** if $S - c$ is disconnected, and **non-separating** if $S - c$ is connected.

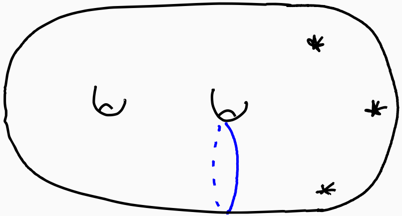
Example:

Studying the mapping class group: curves

A **curve** in S is an embedding of the circle $a\colon S^1 \to S$.

A curve c is **separating** if $S - c$ is disconnected, and **non-separating** if $S - c$ is connected.

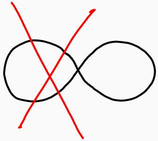
Example:



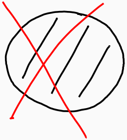
Studying the mapping class group: curves

A **curve** in S is an embedding of the circle $a\colon S^1 \to S$.

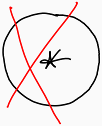
simple



essential

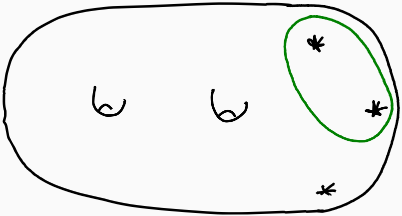


non-peripheral



A curve c is **separating** if $S - c$ is disconnected, and **non-separating** if $S - c$ is connected.

Example:

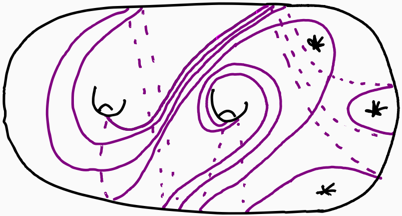


Studying the mapping class group: curves

A **curve** in S is an embedding of the circle $a\colon S^1 \to S$.

A curve c is **separating** if $S - c$ is disconnected, and **non-separating** if $S - c$ is connected.

Example:

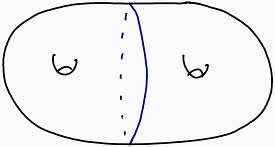


Studying the mapping class group: curves

We typically consider curves up to **isotopy**. Two curves $a, b\colon S^1 \to S$ are isotopic if there is a homotopy between them so that every intermediate map is an embedding.

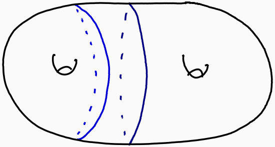
Studying the mapping class group: curves

We typically consider curves up to **isotopy**. Two curves $a, b\colon S^1 \to S$ are isotopic if there is a homotopy between them so that every intermediate map is an embedding.



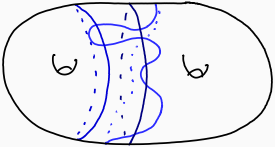
Studying the mapping class group: curves

We typically consider curves up to **isotopy**. Two curves $a, b \colon S^1 \to S$ are isotopic if there is a homotopy between them so that every intermediate map is an embedding.



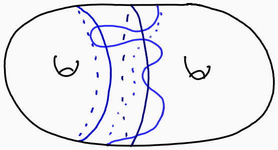
Studying the mapping class group: curves

We typically consider curves up to **isotopy**. Two curves $a, b\colon S^1 \to S$ are isotopic if there is a homotopy between them so that every intermediate map is an embedding.



Studying the mapping class group: curves

We typically consider curves up to **isotopy**. Two curves $a, b\colon S^1 \to S$ are isotopic if there is a homotopy between them so that every intermediate map is an embedding.



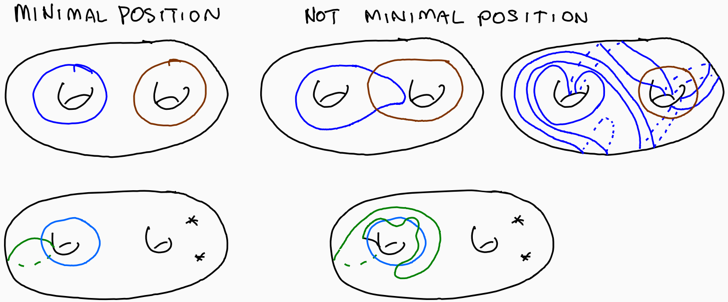
- Homeomorphisms take curves to curves.
- If ϕ_1, ϕ_2 are isotopic homeomorphisms (i.e. representing the same element of $\mathsf{MCG}(S)$) and α_1, α_2 are isotopic curves, then $\phi_1(\alpha_1)$ is isotopic to $\phi_2(\alpha_2)$.

$\to$ There is a well defined action of $\mathsf{MCG}(S)$ on the set of isotopy classes of curves in S.

Minimal position

- c_1 and c_2 (isotopy classes of) curves in S
- γ_1, γ_2 fixed representatives of the isotopy classes c_1, c_2, i.e. actual embeddings of S^1 not considered up to isotopy

We say γ_1 and γ_2 are in **minimal position** if the number of intersections in $\gamma_1 \cap \gamma_2$ is the minimal possible for two curves in the isotopy classes of c_1 and c_2.



Minimal position

- c_1 and c_2 (isotopy classes of) curves in S
- γ_1, γ_2 fixed representatives of the isotopy classes c_1, c_2, i.e. actual embeddings of S^1 not considered up to isotopy

We say γ_1 and γ_2 are in **minimal position** if the number of intersections in $\gamma_1 \cap \gamma_2$ is the minimal possible for two curves in the isotopy classes of c_1 and c_2.

Fact: If $c_1, \ldots, c_k$ is a collection of curves in S, we can realise them so that every pair is simultaneously in minimal position.

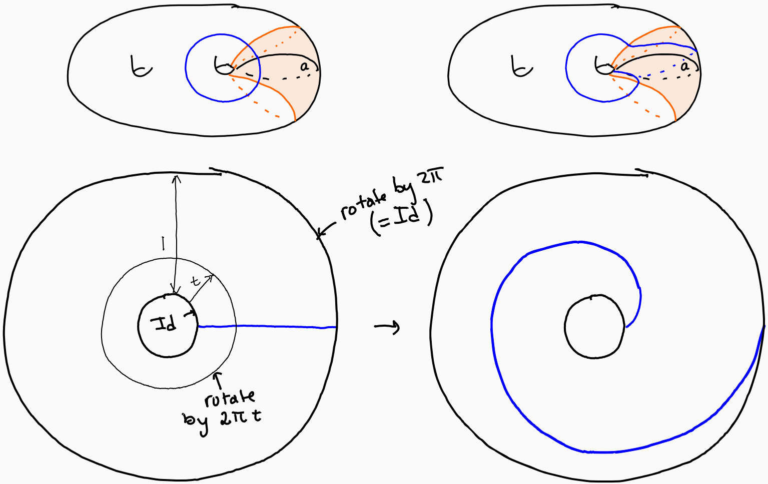
Exercise: 1. Prove this for the torus T^2. (Hint: we can realise T^2 as a quotient of the Euclidean plane, and in each isotopy class of curves there is a representative which is a straight line.)

2. Try to prove for surfaces of negative Euler characteristic using the fact that these admit a hyperbolic metric.

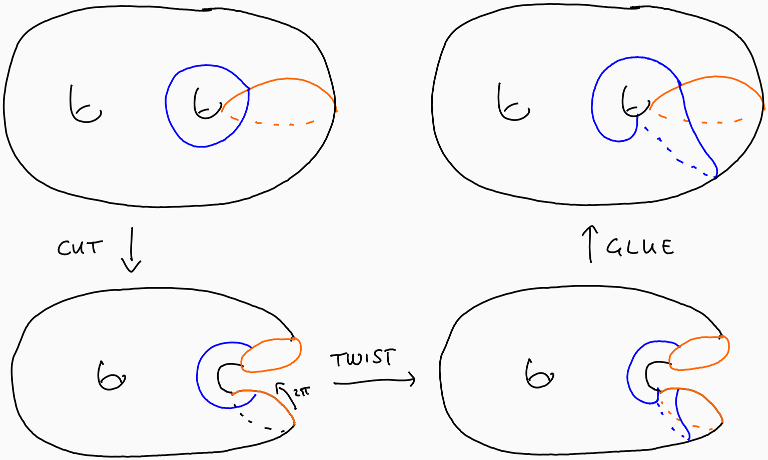
Dehn twists: the building blocks of the mapping class group

Next week we will see how to generate MCG(S) using **Dehn twists**.

To define a Dehn twist about a curve a, we consider an annular neighbourhood of a.



Dehn twists: the building blocks of the mapping class group



For each curve, we have a left Dehn twist and a right Dehn twist, and these are inverses of each other.

When is a mapping class trivial?

The identity element $\mathrm{Id} \in \mathrm{MCG}(S)$ is the class of all self-homeomorphisms of S isotopic to the identity homeomorphism.

If $f = \mathrm{Id} \in \mathrm{MCG}(S)$ then $f(a)$ is isotopic to a for every curve a in S.

If we know that f fixes certain curves (up to isotopy), can we guarantee that $f = \mathrm{Id}$?

The Alexander method: set up

Let's try to understand mapping classes of S by cutting S up into smaller pieces.

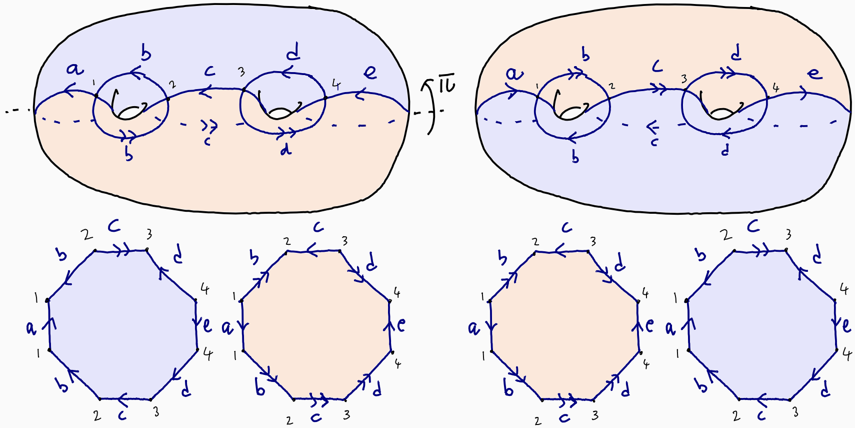
Recall: Let $\mathbf{D}$ be the closed disc and $\mathbf{D}^*$ the once punctured disc. Then $\mathrm{MCG}(\mathbf{D})$ and $\mathrm{MCG}(\mathbf{D}^*)$ are both trivial.

So does this mean that if we add enough curves to cut S into discs and once punctured discs then a mapping class f fixing all of these curves must be trivial?

Well, not quite: f could still permute or rotate the discs.

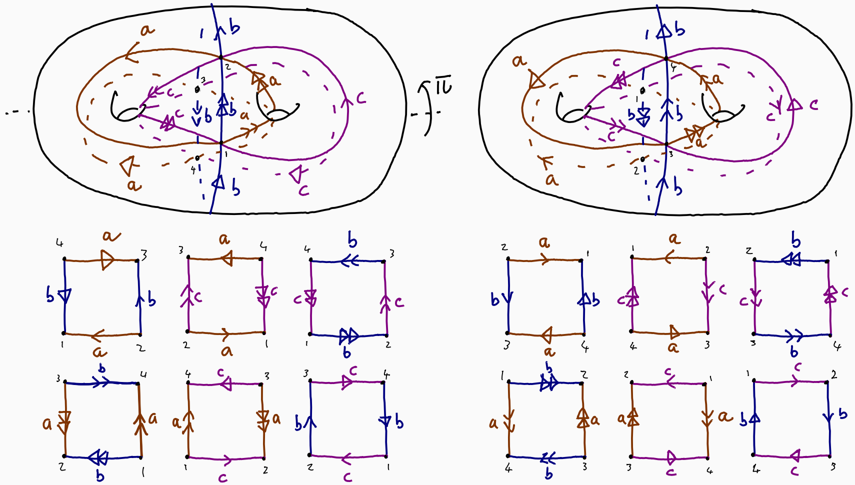
Let's see some examples.

The Alexander method: set up



The curves are preserved but the two discs swap places.

The Alexander method: set up



Each disc (square) is preserved, **but** each is rotated by a half turn.

The Alexander method

Let $c_1, \ldots, c_n$ be distinct oriented curves in S.

Assume $c_1, \ldots, c_n$ are realised in minimal position and let $\Gamma = \bigcup_i c_i$. This is an oriented graph in S. Also assume:

- Γ cuts S into a disjoint union of discs and once punctured discs
- for any distinct i, j, k, one of $c_i \cap c_j$, $c_j \cap c_k$, $c_k \cap c_i$ is empty

  $\rightarrow$ we can realise Γ in a canonical way (up to isotopy)

Let $f \in \mathrm{MCG}(S)$ and suppose that f preserves the collection of curves $c_1, \ldots, c_n$ as a set. Then after possibly applying an isotopy, f preserves the graph Γ, and induces a graph automorphism $f_* \colon \Gamma \to \Gamma$.

Remark: There is something to check here. Namely, we are given that a representative homeomorphism ϕ of the mapping class f takes each c_i to a curve isotopic to some c_j. But we need that there is a single isotopy that works for all c_i at once, so that we can take $\phi(\Gamma)$ to Γ by an isotopy.

The Alexander method

Let $c_1, \ldots, c_n$ be distinct oriented curves in S.

Assume $c_1, \ldots, c_n$ are realised in minimal position and let $\Gamma = \bigcup_i c_i$. This is an oriented graph in S. Also assume:

- Γ cuts S into a disjoint union of discs and once punctured discs
- for any distinct i, j, k, one of $c_i \cap c_j$, $c_j \cap c_k$, $c_k \cap c_i$ is empty
  $\rightarrow$ we can realise Γ in a canonical way (up to isotopy)

Let $f \in \mathrm{MCG}(S)$ and suppose that f preserves the collection of curves $c_1, \ldots, c_n$ as a set. Then after possibly applying an isotopy, f preserves the graph Γ, and induces a graph automorphism $f_* \colon \Gamma \to \Gamma$.

1. If f_* is the identity, i.e. preserving each each edge of Γ with orientation, then $f = \mathrm{Id} \in \mathrm{MCG}(S)$.
2. The set $\{f \in \mathrm{MCG}(S) \mid f \text{ preserves } \bigcup c_i\}$ is a finite group. In particular, any f preserving the set of c_i has finite order.

The Alexander method

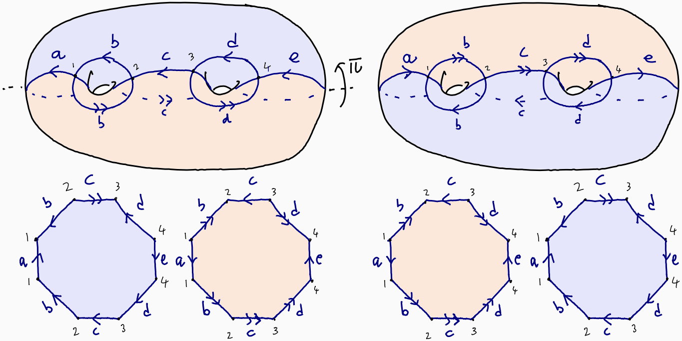
Exercise:

1. (Part of the proof of item 1.) Let $c_1, \ldots, c_n$ be a collection of curves as in the statement of the Alexander method. Suppose that ϕ is a homeomorphism of S that acts as the identity on $\bigcup c_i$ (actually fixing the curves pointwise, not up to isotopy). Use the fact that $\mathrm{MCG}(\mathbf{D})$ and $\mathrm{MCG}(\mathbf{D}^*)$ are trivial to deduce that ϕ is isotopic to the identity.
2. Use item 1. to prove that the map

$$\{f \in \mathrm{MCG}(S) \mid f \text{ preserves } \bigcup c_i\} \to \mathrm{Aut}(\Gamma)$$

is injective, and deduce item 2.

The Alexander method: back to first example



The oriented graph is preserved, but the individual edges are not.

Alexander method → this mapping class has finite order: indeed we can see it has order 2.

If every edge of the graph was preserved with orientation, then we would have the identity.

Example: relations in the mapping class group

We can use the Alexander method to check relations in the mapping class group.

Alexander method $\rightarrow$ we only need to check the relation on a **finite** collection of curves, and the graph they form.

NB: We apply mapping classes from right to left.

NB: I will use the convention that a positive (not inverse) Dehn twist will twist left in the picture.

Example: The "braid relation". If a and b are two curves intersecting once, and T_a, T_b are the Dehn twists about a, b respectively, then $T_aT_bT_a = T_bT_aT_b$.

Aside: "change of coordinates"

Example: Braid relation. a, b intersect once $\rightarrow T_a T_b T_a = T_b T_a T_b$.

Crucial observation: We don't need to check every pair a, b of curves intersecting once.

Claim: for any two pairs a, b and a', b' with each pair intersecting once, there exists $f \in \mathrm{MCG}(S)$ so that $f(a) = a'$ and $f(b) = b'$.

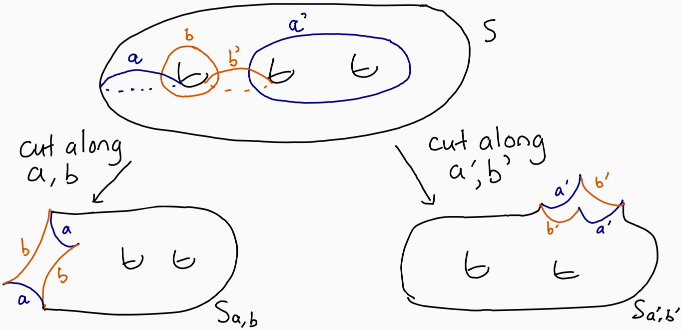
Assuming the claim, we have:

- $T_{a'} = T_{f(a)} = f\, T_a f^{-1}$
- $T_{b'} = T_{f(b)} = f\, T_b f^{-1}$

And hence $T_a T_b T_a = T_b T_a T_b \iff T_{a'} T_{b'} T_{a'} = T_{b'} T_{a'} T_{b'}$.

Exercise: Use the definition of Dehn twist in terms of an annular neighbourhood of a to check $T_{f(a)} = f\, T_a f^{-1}$.

Aside: "change of coordinates"

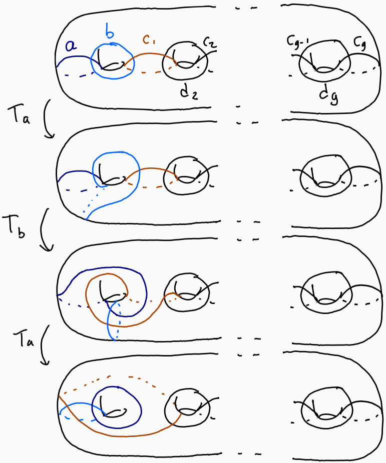


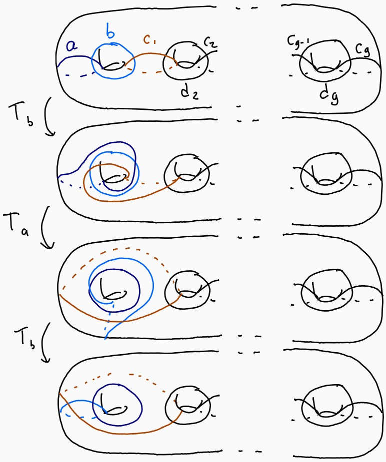
- Classification of surfaces → there exists a homeomorphism $\phi\colon S_{a,b} \to S_{a',b'}$
- Isotope ϕ so that it
  - takes arcs of a to arcs of a'
  - takes arcs of b to arcs of b'
  - respects how the arcs are glued up
- glue back together, and we have a homeo. taking a, b to a', b'

Example: relations in the mapping class group

Example: Braid relation. a, b intersect once $\rightarrow T_a T_b T_a = T_b T_a T_b$.

For simplicity assume S has no punctures. Fix a set of curves cutting S into discs, with no three curves pairwise intersecting.





Example: relations in the mapping class group

Exercises:

1. Make an oriented graph Γ from the curves in the braid relation example and check what happens to this when we do the twists. Check the induced automorphisms of Γ are the same for $T_a T_b T_a$ and $T_b T_a T_b$.
2. Check the case where S has genus at least 1 and might have punctures (add some more curves to satisfy the hypotheses of the Alexander method).
3. Convince yourself that two curves on a surface of genus 0 cannot intersect exactly once – so there is nothing to check here.