An Introduction to Mapping Class Groups 2

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26 October 2020

IHES

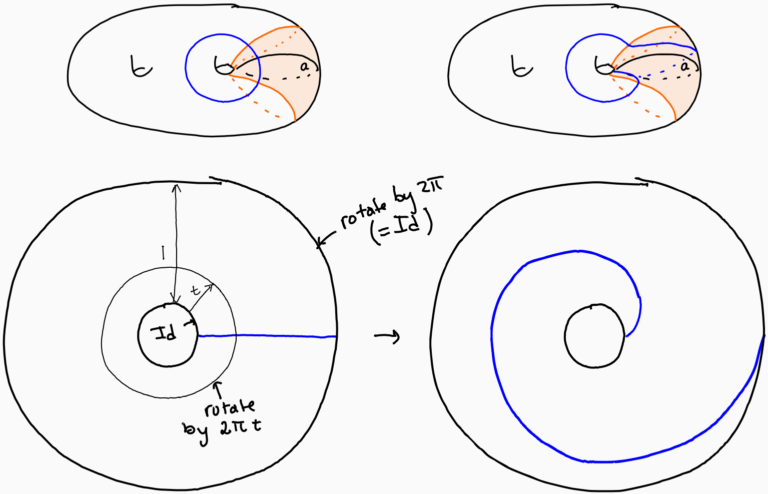
Recall: the mapping class group

$S = S_{g,p}$ a connected, oriented, finite type surface

$\mathrm{MCG}(S) = \{$orientation-preserving homeomorphisms $S \to S\}/$isotopy

Recall: Dehn twists

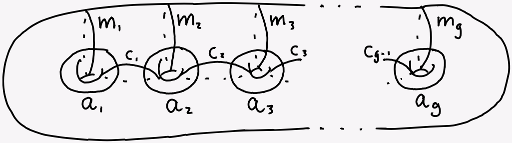
To define a Dehn twist about a curve *a*, we consider an annular neighbourhood of *a*.



Generating the mapping class group

Theorem (Lickorish generators for $\mathrm{MCG}(S_{g,0})$)

For $g \geq 1$ the mapping class group $\mathrm{MCG}(S_{g,0})$ is generated by Dehn twists about the $3g-1$ curves pictured.



To sketch a proof of this, we will introduce two important tools in the study of the mapping class group:

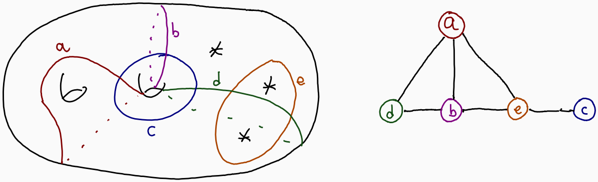
- graphs of curves
- the Birman exact sequence

Notation: For a curve b, we will write T_b for the (left) Dehn twist about b.

The curve graph

The curve graph $\mathcal{C}(S)$ of S has:

- a vertex for every isotopy class of curves in S
- an edge joining two vertices if the isotopy classes have disjoint representative curves



This is a (locally infinite) simplicial complex, and we can make it a metric space by giving each edge length 1.

MCG(S) acts on the curve graph by

- simplicial automorphisms
- isometries

The curve graph is connected

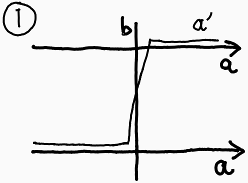
Proposition

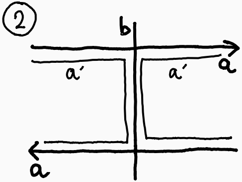
Let S be such that it contains pairs of disjoint curves (i.e. S is not $S_{0,p}$, $p \leq 4$ or $S_{1,p}$, $p \leq 1$). Then the curve graph of S is connected.

Sketch of proof: Let a, b be two curves, arranged to intersect minimally. Let a, b intersect n times. We use an induction on n to prove that there is a path from a to b.

Orient a and consider two points of $a \cap b$ consecutive along b.

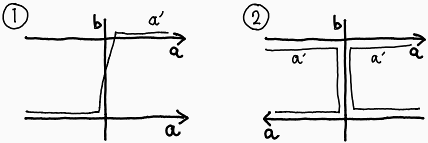
We use a **curve surgery**.





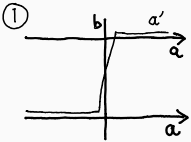
The curve graph is connected

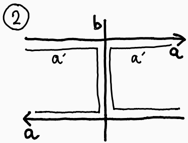
Exercise:



- Base case of induction: if a and b intersect at most once then they are connected by a path in $\mathcal{C}(S)$. (Hint: consider the boundary of a small regular neighbourhood of $a \cup b$ in S.)
- Case 1
  - a' is a single simple closed curve.
  - $a' \cap b$ has fewer points than $a \cap b$ (in minimal position).
  - a and a' intersect exactly once.
  - a' is essential and non-peripheral. (Hint: use the preceding item.)

The curve graph is connected





Exercise:

- Case 2
  - a' is a pair of simple closed curves.
  - $a' \cap b$ has fewer points than $a \cap b$.
  - a and a' are disjoint.
  - **Either** at least one component of a' is essential and non-peripheral, **or** there is another pair of intersections where we will get an essential non-peripheral curve using these surgeries (Hint: if this was not the case, what would the options be for S?)

Properties of the curve graph

Exercise: (assume S admits pairs of disjoint non-isotopic curves, i.e, S is not $S_{0,p}$, $p \leq 4$ or $S_{1,p}$, $p \leq 1$)

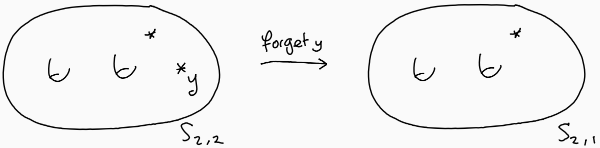
1. Convince yourself that each vertex of the curve graph $\mathcal{C}(S)$ has infinite degree.
2. Prove that the quotient of $\mathcal{C}(S)$ by the natural action of the mapping class group is a finite graph. (Hint: use the "change of coordinates" idea from the end of the last lecture.)

The Birman exact sequence: the forgetful map

For this, we will consider the **pure mapping class group** PMCG(S).

This is the subgroup of MCG(S) that fixes each puncture of S.

Let $S = S_{g,p}$ with $p \geq 1$ and choose a puncture y. If we "forget" the puncture y we get the surface $S_{g,p-1}$.

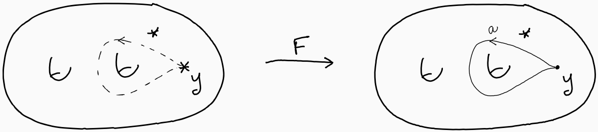


Suppose that f is a mapping class in PMCG(S). Since f is a pure mapping class, it does not swap y with any other puncture of S. Hence f still makes sense when we forget y, and we get an element of PMCG($S_{g,p-1}$).

The Birman exact sequence: kernel of the forgetful map

Let $F\colon S_{g,p} \to S_{g,p-1}$ be the map forgetting the puncture y and $F_*\colon \mathrm{PMCG}(S_{g,p}) \to \mathrm{PMCG}(S_{g,p-1})$ be the induced map on pure mapping class groups.

What does the kernel of F_* look like?

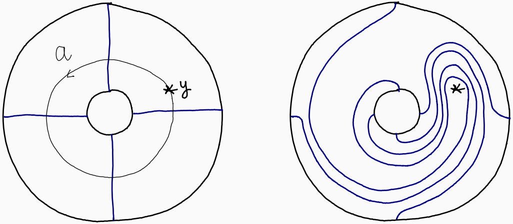


Consider the fundamental group of $S_{g,p-1}$ based at the place where the puncture y is in $S_{g,p}$.

Let a be a loop in $\pi_1(S_{g,p-1}, y)$. We are going to do a **point push** along the loop a.

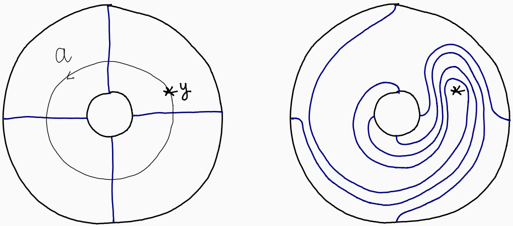
The Birman exact sequence

A point push involves dragging the puncture y around the loop a until it gets back to where it started.



Exercise: Suppose a is a simple (i.e. embedded) loop. If c is the inner boundary component in the picture above and d is the outer boundary component, then the point push Push(a) $= T_c T_d^{-1}$.

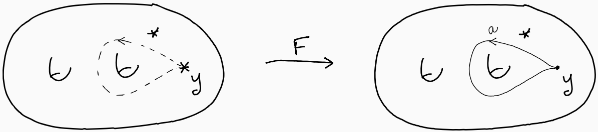
The Birman exact sequence: point pushing



The point push along a is a non-trivial mapping class of $S_{g,p}$ (as long as a is not homotopically trivial).

But when we forget the puncture y and go to $S_{g,p-1}$, the point push is isotopic to the identity, and hence is trivial in $\mathrm{PMCG}(S_{g,p-1})$.

The Birman exact sequence

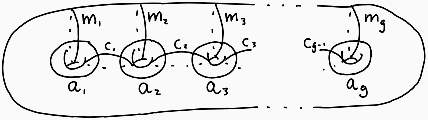


Theorem (Birman Exact Sequence)

Let $F\colon S_{g,p} \to S_{g,p-1}$ be the map forgetting the puncture y and F_* the induced map on pure mapping class groups. We have a short exact sequence:

$$1 \to \pi_1(S_{g,p-1}, y) \xrightarrow{\text{Push}} \mathrm{PMCG}(S_{g,p}) \xrightarrow{F_*} \mathrm{PMCG}(S_{g,p-1}) \to 1.$$

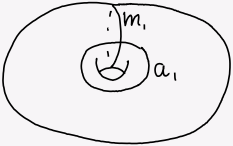
Constructing the Lickorish generating set



Now we can start proving that the Lickorish twists generate $\mathrm{MCG}(S_g)$, where $S_g = S_{g,0}$. (NB: the proof will follow the one in *A Primer on Mapping Class Groups*.) We will use an induction on g, starting from:

Fact

$\mathrm{MCG}(S_{1,0})$ is generated by Dehn twists about the following two curves:



Constructing the Lickorish generating set

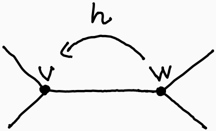
Assume $g \geq 2$. We are going to use the following fact to reduce checking the generating set for $\mathrm{MCG}(S)$ to a lower genus.

Fact

Suppose

- X connected simplicial graph (no loops or double edges)
- G acts on X by simplicial automorphisms
- G acts transitively on ordered pairs of adjacent vertices of X

Let v, w be adjacent vertices of X and let $h \in G$ so that $h(w) = v$. Then $G = \langle h, \mathrm{Stab}_G(v) \rangle$, where $\mathrm{Stab}_G(v) = \{g \in G \mid g(v) = v\}$.

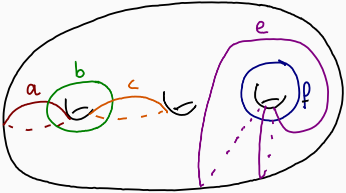


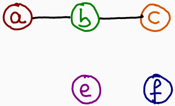
Constructing the Lickorish generating set

We will apply the fact from the previous slide to MCG(S), using a variation of the curve graph of S.

Let $\mathcal{N}(S)$ be the graph with:

- a vertex for every isotopy class of **non-separating curves**
- an edge between two vertices if the curves intersect exactly once





Exercise: Prove that $\mathcal{N}(S)$ is connected, using a similar surgery argument to that for the curve graph. (Hint: for "Case 2", try looking at three consecutive intersections instead of just two.)

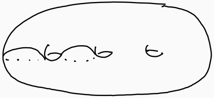
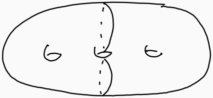
Constructing the Lickorish generating set

Let $\mathcal{N}(S)$ be the graph with:

- a vertex for every isotopy class of **non-separating curves**
- an edge between two vertices if the curves intersect exactly once

Remark: Why $\mathcal{N}(S)$ instead of the normal curve graph, or the graph of non-separating curves with edges for disjointness?

Well, to apply the Fact, we want the mapping class group to act **transitively** on edges. Last week we saw that $\mathrm{MCG}(S)$ acts transitively on pairs of non-separating curves intersecting exactly once ("change of coordinates"). This is not true for pairs of disjoint (non-separating) curves.

Constructing the Lickorish generating set

Fact

Suppose

- X connected simplicial graph (no loops or double edges)
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- G acts transitively on ordered pairs of adjacent vertices of X

Let v, w be adjacent vertices of X and let $h \in G$ so that $h(w) = v$. Then $G = \langle h, \mathrm{Stab}_G(v) \rangle$, where $\mathrm{Stab}_G(v) = \{g \in G \mid g(v) = v\}$.

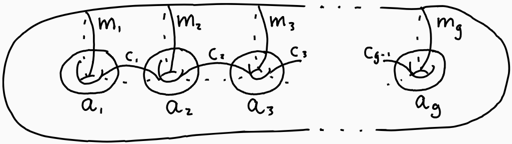
$G = \mathrm{MCG}(S)$, $X = \mathcal{N}(S)$ satisfy the hypotheses.

Let a, m be adjacent vertices of $\mathcal{N}(S)$, i.e. non-separating curves intersecting exactly once.

Exercise: $T_a T_m(a) = m$.

Hence $\mathrm{MCG}(S)$ is generated by T_a, T_m and the stabiliser of m.

Constructing the Lickorish generating set



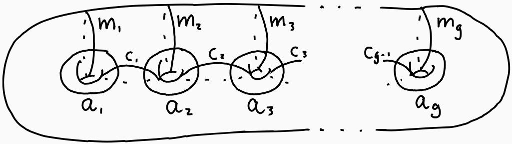
We have just seen that if a and m are two curves intersecting exactly once, then $\mathrm{MCG}(S)$ is generated by T_a, T_m and the stabiliser of m.

Let $a = a_g$ and $m = m_g$. The twists around a_g and m_g are in the set we are trying to prove generate $\mathrm{MCG}(S_g)$.

So it remains to prove: $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(m_g)$ is generated by the twists around the curves in the picture.

We want to reduce this to a question about $\mathrm{MCG}(S_{g-1})$.

Constructing the Lickorish generating set



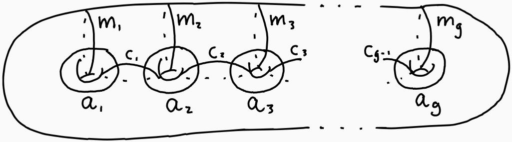
Instead of looking at the full stabiliser of m_g, it will be convenient to stabilise m_g, **with orientation**. We denote the group that does this by $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g)$.

$\mathrm{Stab}_{\mathrm{MCG}(S_g)}(m_g)$ is generated by $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g)$, plus an element that reverses the orientation of m_g.

Exercise: $T_{a_g} T_{m_g}^2 T_{a_g}$ preserves m_g but reverses its orientation.

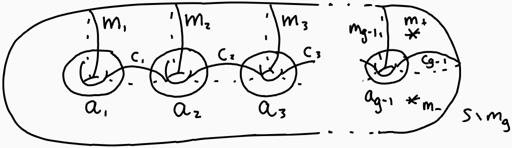
This element can be formed from our set of Lickorish twists, so it suffices to check that $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g)$ is generated by these twists.

Constructing the Lickorish generating set



Lemma: $\mathrm{Stab}_{\mathrm{MCG}(S_g)}(\vec{m}_g) = \mathrm{PMCG}(S - m_g) \times \langle T_{m_g} \rangle$

Constructing the Lickorish generating set



$$S - m_g \cong S_{g-1,2}$$

We have reduced the genus. The induction hypothesis tells us that if we forget the two punctures m_+ and m_-, $\mathrm{MCG}(S_{g-1})$ is generated by $a_1, \ldots, a_{g-1}$, $m_1, \ldots, m_{g-1}$, $c_1, \ldots c_{g-2}$.

So now we want to compare $\mathrm{PMCG}(S_{g-1,2})$ with $\mathrm{MCG}(S_{g-1}) = \mathrm{PMCG}(S_{g-1})$.

Constructing the Lickorish generating set

We can use the Birman exact sequence twice:

$$1 \to \pi_1(S_{g-1,1}, m_-) \xrightarrow{\text{Push}} \text{PMCG}(S_{g-1,2}) \xrightarrow{F_*} \text{PMCG}(S_{g-1,1}) \to 1$$

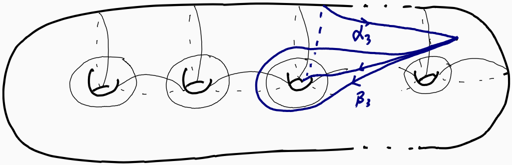
$$1 \to \pi_1(S_{g-1}, m_+) \xrightarrow{\text{Push}} \text{PMCG}(S_{g-1,1}) \xrightarrow{F_*} \text{PMCG}(S_{g-1}) \to 1$$

Constructing the Lickorish generating set

$$1 \to \pi_1(S_{g-1,1}, m_-) \xrightarrow{\text{Push}} \mathrm{PMCG}(S_{g-1,2}) \xrightarrow{F_*} \mathrm{PMCG}(S_{g-1,1}) \to 1$$

$$1 \to \pi_1(S_{g-1}, m_+) \xrightarrow{\text{Push}} \mathrm{PMCG}(S_{g-1,1}) \xrightarrow{F_*} \mathrm{PMCG}(S_{g-1}) \to 1$$

What needs to be done to finish the proof is to carefully choose generating sets for $\pi_1(S_{g-1}, m_+)$ and $\pi_1(S_{g-1,1}, m_-)$ and to prove that the point pushes around each of the loops in the generating set can be written as a product of Lickorish twists.



Other applications of curve graphs: homological properties of mapping class groups

Curve graph $\mathcal{C}(S)$ (assume S admits pairs of disjoint curves):

- a vertex for every isotopy class of curves in S
- an edge joining two vertices if curves are disjoint

We can make the curve graph into a flag complex by adding a n-simplex for every $(n+1)$-clique in the graph. In fact this is the original definition of the **curve complex** but for many purposes we just think about the 1-skeleton.

Harer used the curve complex to prove various results about homological properties of the mapping class groups[1][2].

[1] *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. Math. **121** (1985)

[2] *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986)

Other applications of curve graphs: algebraic properties of mapping class groups

Ivanov showed[3] (for genus at least 2; Korkmaz and Luo proved the remaining cases) that the group of simplicial automorphisms of the curve graph $\mathcal{C}(S)$ is the **extended mapping class group** $\mathrm{MCG}^{\pm}(S)$. This is defined just like the mapping class group except that we allow orientation-reversing homeomorphisms.

A consequence is that the automorphism group of $\mathrm{MCG}(S)$ is equal to $\mathrm{MCG}^{\pm}(S)$ and every automorphism $\mathrm{MCG}(S) \to \mathrm{MCG}(S)$ is given by conjugation. In fact any isomorphism $\Gamma_1 \to \Gamma_2$, where Γ_1, Γ_2 are finite index subgroups of $\mathrm{MCG}(S)$, is given by conjugation by some element of $\mathrm{MCG}^{\pm}(S)$.

[3] *Automorphism of complexes of curves and of Teichmüller spaces*, Int. Math. Res. Not. **14** (1997)

Other applications of curve graphs: algebraic properties of mapping class groups

Different variations of the curve graph have been used to prove similar results for different normal subgroups of the mapping class group, notably in work of Brendle and Margalit[4].

[4] *Normal subgroups of mapping class groups and the metaconjecture of Ivanov*, J. Amer. Math. Soc. **32** (2019)

Other applications of curve graphs: large scale geometry of mapping class groups

To make $\mathcal{C}(S)$ into a metric space, we give each edge length 1.

Theorem (Masur–Minsky[5])

$\mathcal{C}(S)$ is a δ-hyperbolic infinite diameter metric space.

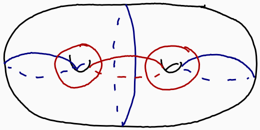
MCG(S) acts on $\mathcal{C}(S)$ by isometries. This action is not properly discontinuous (the infinite cyclic group generated by a Dehn twist about a fixes the vertex a).

[5] *Geometry of the complex of curves I: Hyperbolicity*, Invent. Math. **138** (1999)

Other applications of curve graphs: large scale geometry of mapping class groups

Masur and Minsky later showed[6] that the word metric on the mapping class group of S can be estimated by a sum of distances in curve graphs of subsurfaces of S.

To compare distances in $\mathrm{MCG}(S)$ with distances in curve graphs, we model $\mathrm{MCG}(S)$ by a graph $\mathcal{M}(S)$ whose vertices are collections of curves that cut S into discs and once punctured discs, with appropriate edges. The Alexander method can be used to show that $\mathrm{MCG}(S)$ acts properly discontinuously and cocompactly – so that $\mathcal{M}(S)$ is quasi-isometric to $\mathrm{MCG}(S)$.



[6] *Geometry of the complex of curves II: Hierarchical structure*, Geom. Funct. Anal. **10** (2000)