Masters Thesis
Topological Gauge Theories and Volumes of the Moduli Space of Flat Connections

Campbell Wheeler

Supervisor: Paul Norbury

The School of Mathematics and Statistics
The University of Melbourne

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Contents

Notation

This list is a guide to the notation used throughout this thesis. It is only a guide and the symbols used will hopefully be clear from the context. Also note that there are some double ups of notation however they are often in separate sections or quite clear from context.

$[-]$ - Equivalence class
M, N - Smooth Manifold
K - Sub-manifold
Σ - Surface
P, Q - Principle Bundle
H - Horizontal Subspace
V - Vertical Subspace
$\mathcal{G}_P$ - Gauge Group of P
H, K - Elements of the Gauge Group
G - Lie Group (usually Compact Connected Semisimple)
ω_G - Maurer-Cartan Form
$\mathfrak{g}$ - Lie Algebra
κ - Killing Form
$\mathfrak{h}$ - Cartan Sub-algebra
F - Weyl Denominator (Following [Wit91])
G - Group
e - $e \in G$ denotes the Identity Element.
$G/Ad(G)$ - Conjugacy Classes of G
R_g - For right G action on P the map $R_g(p) = p \cdot g$
L_g - For left G action on P the map $L_g(p) = p \cdot g$
L_p - Fibre map $L_p(g) = p \cdot g$
$\widehat{\mathfrak{g}}$ - Affine Lie Algebra
X, Y - Topological Space
B - Base of some Fibre Bundle
F - Fibre of a Fibre Bundle
E - Total Space of a Fibre Bundle
$\mathfrak{g}_P$ - Adjoint Bundle associated to P
π_P - Projection of the Adjoint Bundle
$\eta_{P,\kappa}$ - Metric on the Adjoint bundle induced by κ
π - Projection Map
C^∞ - Smooth
PWC^∞ - Piecewise Smooth

Γ - Sections
$\mathfrak{X}$ - Vector Fields
Ω - Differential Forms
U, V - Open Subset
K - Compact Subset
V - Vector Space
$\mathbb{F}$ - Field
ω - Symplectic Form
$\Sigma_{g,n}$ - Surface of Genus g with n Boundary Circles/Punctures
$\mathcal{M}_{g,n}$ - Moduli Space of Curves on $\Sigma_{g,n}$
$\mathcal{R}_{G,g,n}$ - Moduli Space of Flat Connections on $\Sigma_{g,n}$ with fibre G ([Wei98])
$\mathcal{R}_{M,G}$ - Moduli Space of Flat Connections on M with fibre G
$\mathcal{A}_P$ - Space of Connections of P
A, B - Connections
ω_A - Connection 1-form associated to $A \in \mathcal{A}_P$
F_A - Curvature of A
φ_{H_A} - Horizontal Projection associated to $A \in \mathcal{A}_P$
φ_{V_A} - Vertical Projection associated to $A \in \mathcal{A}_P$
g^{AB} - Atiyah-Bott Metric
$\omega^{AB,P,\kappa}$ - Atiyah-Bott Symplectic Form
$Trin$ - Trinion/Pair of Pants Decomposition
γ - Path or Curve $\gamma : [0, 1] \to -$
$\sqcup$ - Disjoint Union
$Stab$ - Stabiliser
Orb - Orbit
$i = \sqrt{-1}$ or $i \in \mathbb{Z}_{>0}$
Z - TQFT Functor
Z_G - Dijkgraaf-Witten TQFT Functor (in $(1 + 1)$-dimensions with Lie Groups as well)

How to Read

There are different ways to read this document depending on your prior knowledge. The thesis assumes some basic knowledge in the theory of smooth manifolds and Lie groups. There is a very brief exhibition of these theories in Appendix A and Appendix B.

The Novice: For people with little background in the theory of principle bundles and connections it is recommended that one first reads Appendix C, Appendix D and Appendix E and then proceeds to Chapter 1.

For people with little background in the theory of symplectic geometry it is recommended that one first reads Appendix F and then proceeds to Chapter 2 and Chapter 3.

For people with little background in topological quantum field theory it is recommended that one first reads Appendix G then proceeds to Chapter 4.

Those with a Working Knowledge: For people with a working knowledge of the theory of principle bundles, connections, symplectic geometry and topological quantum field theory the thesis can simply be read Chapter 1 through to Chapter 4.

The Expert: For the expert there are explicit volume calculations using symplectic geometry in Chapter 3, as well as a discussion on recursions for volumes similar to calculations in topological quantum field theory. In Chapter 4 there is a description of trivial Dijkgraaf-Witten topological quantum field theory that leads to generalisations in $(1 + 1)$-dimensions to Lie groups related to the volumes of the moduli space of flat connections.

Introduction

Gauge theories in physics are special kinds of field theories. They are defined to have, what physicists call, local symmetries. That is, the symmetries of the theory are given by smooth maps from its space-time into some Lie group. The topological aspects of these theories have led to many interesting invariants of low dimensional manifolds as described in [Ati90].

The prototypical example of a gauge theory is the theory of electro-magnetism with space-time given by $\mathbb{R}^3 \times \mathbb{R}^1$. The physical observables of the theory such as the electro-magnetic field are captured in the gauge field $A : \mathbb{R}^3 \times \mathbb{R}^1 \to \mathbb{R}^3 \times \mathbb{R}^1$. The information of the observables is contained in the tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This can be expressed using differential forms as $A = a_0 dx_0 + a_1 dx_1 + a_2 dx_2 + a_3 dx_3$ and $F = dA$. Maxwell's equations of motion then take the form

$$dF = 0 \quad d^* F = 0$$

For $f : \mathbb{R}^3 \times \mathbb{R}^1 \to U(1)$ notice that $A + f^{-1}df$ and A give rise to the same physical observables as $d(A + f^{-1}df) = dA + d(f^{-1}df) = dA$. We call $f : \mathbb{R}^3 \times \mathbb{R}^1 \to U(1)$ a $U(1)$-gauge symmetry. This shows that there is a large redundancy in the description of the system through the gauge field A. This can be exploited when calculating the dynamics of the system.

The symmetry described by f maps into $U(1)$ and we call electro-magnetism a $U(1)$ gauge theory. The map $A \mapsto A + f^{-1}df$ is called a gauge transformation. The key to generalising this is noting that A in fact defines what is called a $U(1)$ connection on $\mathbb{R}^3 \times \mathbb{R}^1$. In particular if we let $\mathfrak{u}(1)$ be the Lie algebra of $U(1)$ then $A = a_0 dx_0 + ... a_3 dx_3$ where $a_\mu : \mathbb{R}^3 \times \mathbb{R}^1 \to \mathfrak{u}(1) \cong i\mathbb{R} \cong \mathbb{R}$.

To generalise electro-magnetism or gauge theory with a $U(1)$ symmetry to gauge theory with a Lie group G symmetry we now take $A : \mathbb{R}^3 \times \mathbb{R}^1 \to \mathfrak{g}^3 \times \mathfrak{g}^1$. We then take $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu, a_\nu]$. This can be expressed using differential forms as $A = a_0 dx_0 + ... + a_3 dx_3$ and $F_A = d_A A$. The generalisation of Maxwell's equations is then

$$d_A F_A = 0 \quad d_A^* F_A = 0$$

where d_A is the covariant derivative. The set $\mathcal{A}_{flat} = \{A : F_A = 0\}$ clearly satisfies the above equations. Moreover, we can define the gauge theory through a Lagrangian called the Yang-Mills functional, which is described in [AB83]. The set $\mathcal{A}_{flat}$ then gives the minimal solutions to the equations of motion.

We can generalise further by considering topologically interesting space-times of various dimensions with interesting metrics. To do this, one defines connections. A connection on $\mathbb{R}^3 \times \mathbb{R}^1 \times G$ corresponds to the gauge potential A. We can generalise the definition of connection further to principle bundles $P \to M$ with fibre G.

In this general viewpoint, elements of the set of solutions $\mathcal{A}_{flat}$ correspond to fixing local product structures, or trivialisations, of the bundle or in the previous example $\mathbb{R}^3 \times \mathbb{R}^1 \times G$.

In the first chapter of this thesis we will classify the set of solutions $\mathcal{A}_{flat}$ up to gauge equivalence. Moreover, we will endow the set of solutions with a topology. This solution space will remarkably be a finite dimensional space with a dense set of smooth points. The solution space will be called the moduli space of flat connections.

For example, the solution space or moduli space of flat connections over a genus g surface with $U(1)$ local symmetries is $U(1)^{2g}$. In section 1.2.1 we will prove that for arbitrary Lie group G the solution space or moduli space of flat connections over the circle S^1 with G local symmetries is given by the conjugacy classes of G which we denote $G/Ad(G)$.

An important observation is that understanding the solution space or the moduli space of flat connections can tell us interesting topological aspects of the space-time. This is the underlying theme of topological quantum field theory.

To study the moduli space we endow it with additional structures. In any dimensional space-time with a fixed Riemannian metric the moduli space can be given a Riemannian metric. Restricting our attention to surfaces the moduli space can be given a symplectic form which with the Riemannian metrics form Kähler structures.

In the second chapter of this thesis we will describe the construction of the Riemannian metric and symplectic form given in [AB83]. Given a path through the space-time, we can define special functions from the moduli space and for surfaces Goldman calculates the Hamiltonian flows and Poisson brackets of these functions in [Gol86].

Using these functions and some results in the theory of symplectic manifolds, in the third chapter of this thesis we calculate the symplectic volume of the moduli space over a surface with $SU(2)$ local symmetries. We then describe the results of Witten in [Wit91] for general Lie groups G. The solution space will generally be uncountable. The volume gives an analogy for counting the number of solutions.

So far we have described gauge theories with a Lie group G local symmetries. In the fourth chapter of this thesis we consider gauge theories with a finite group G local symmetries as done in [DW90]. The theory of finite gauge symmetries corresponds to the theory of covering spaces.

The moduli space or solution space is finite for finite gauge symmetries. Therefore, we can count the solutions. Moreover, we can define a topological quantum field theory that counts the solutions which corresponds to counting G-covers. When one considers $(1+1)$-dimensions, we find that the number of G-covers of a genus g surface is given by

$$\#G^{2g-2} \sum_{\alpha \in irred(G)} \frac{1}{\dim(\alpha)^{2g-2}}$$

The volume of the solution space or moduli space for local symmetries given by a Lie group G described in theorem 3.2.1 is given by the series

$$Z(G)Vol(G)^{2g-2} \sum_{\alpha \in irred(G)} \frac{1}{\dim(\alpha)^{2g-2}}$$

Using this observation in section 4.2 we describe a well defined topological quantum field theory that calculates the volume of the moduli space.

Chapter 1

The Moduli Space of Flat Connections

In this chapter, we will describe the fundamental objects needed to define the moduli space of flat connections and its topology. These definitions are of interest as they lay at the cross-section of mathematics and physics. They are the basic tools that we need to study what is sometimes called gauge theory, which presents itself mathematically as the theory of connections on smooth principle bundles. Two good references are [Aud04] and [AB83]. There have been many remarkable results concerning the topology and symplectic geometry of these spaces and notably in [AB83] they find generators for the cohomology ring using Morse theory, the Yang-Mills functional and equivariant cohomology. Symplectic volumes are calculated in [Wit91] [JW94]. Building on this work, intersection numbers of these cohomology classes have been calculated using a variety of methods notably in [Tha95] [Wit92] [JK98]. Not only do they have a remarkable theory for themselves but connections also provide the tools to define some interesting topological quantum field theories such as Chern-Simons theory. This has led to new knot invariants and more generally new invariants of 3-manifolds. An introduction to this theory can be found in [Ati90].

We have supplied a reasonably self contained account of the very basics of the differential topology and Lie theory needed in the appendices. The sections of particular interest are Appendix C, Appendix D and Appendix E. For the novice, this and the references found there would an important starting point. The masters thesis of Michiels [Mic13] also steps through some of the background needed.

1.1 The Moduli Space of Flat Connections and the Gauge Group

The gauge group defines an equivalence relation on the set of flat connections. The set of equivalence classes can be endowed with a topology which allows us to understand various properties of connections and their underlying manifolds. Heuristically, we have an idea of when connections should be close and this will be quantified with a topology. Constructing a space such as the moduli space of flat connections is one method for understanding and creating invariants of manifolds. As mentioned in the introduction of chapter 1, this has had great success over the last few decades at creating new invariants and reinventing old invariants.

1.1.1 The Space of Connections

We will briefly recall the basic structure of the space of connections. This is a small summary of what is covered in Appendix C. For more details see Appendix C.

> **Definition:** (Connection)
> Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $V = ker(\pi_*)$ and $R_g : P \to P$ such that $R_g(p) = p \cdot g$.
>
> If $H \subseteq TP$ is a smooth sub-bundle such that $TP = V \bigoplus H$ and $T_p R_g(H_p) = H_{p \cdot g}$ for all $g \in G$ and $p \in P$ we call H a **connection** on P.

> **Definition:** (Connection 1-forms)
> For $L_p : G \to P$ such that $L_p(g) = p \cdot g$. We say that $\beta \in \Omega^1(P, \mathfrak{g})$ is a **connection 1-form** if β is pseudotensorial and $\beta_p(T_e L_p(v)) = v$, that is for the fundamental vector field associated to v given by v^* we have $\beta(v^*) = v$.

Lemma 1.1.1. *(Lemma C.2.2)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. The following objects are canonically bijective

- *Connections on P.*
- *Smooth linear equivariant vertical projections $\varphi_V : TP \to V$ that is a splitting of the short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, hook] & TP \arrow[r, two heads] \arrow[l, bend left, dashed, "\varphi_V"] & TP/V \arrow[r] \arrow[l, bend left, dashed] & 0 \end{tikzcd}$$

- *Connection 1-forms on P*

> **Definition:** (Adjoint Bundle)(See Appendix C.1)
> Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $\pi_P : \mathfrak{g}_P = P \times_G \mathfrak{g} \to M$ be the **adjoint bundle**.

Lemma 1.1.2. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. The difference inherited by $\Omega^1(P, \mathfrak{g})$ gives the connection 1-forms on P the structure of an affine space modelled on $\Omega^1(M, \mathfrak{g}_P)$.*

> **Definition:** (The Space of Connections)(See section C.3)
> Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $\mathcal{A}_P$ denote the **space of connections** on P. This is an affine space modelled on $\Omega^1(M, \mathfrak{g}_P)$ and if P is trivial then this is canonically identified with the vector space $\Omega^1(M, \mathfrak{g})$.

Notation: Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. We will use the following notation

- $H_A \subseteq V \oplus H_A = TP$ will be the horizontal subspace associated to A
- φ_{V_A} will be the equivariant vertical projection associated to A.
- $id - \varphi_{V_A} = \varphi_{H_A}$ will be the equivariant horizontal projection associated to A
- $\omega_A \in \Omega^1(P, \mathfrak{g})$ will be the connection 1-form associated to A

1.1.2 The Gauge Group

The gauge group corresponds to the set of automorphisms of P that cover the identity. The gauge group therefore ignores the automorphisms of the base manifold. Automorphisms, and therefore in particular the gauge group, has a natural action on the space of connections.

Definition: (Gauge Group)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Define the **gauge group** of $\mathcal{G}_P = \{H \in Aut(P) : \pi \circ H = id\}$.

We can also identify this with the maps $h : P \to G$ such that $h(p \cdot g) = g^{-1}h(p)g$. That is equivariant maps with respect to the conjugate action of G on itself.

This can also be identified with sections of the bundle of groups $\Gamma(Ad(P)) = \Gamma(P \times_G G)$ where the Cartesian product is taken with respect to the conjugate action of G on itself.

Remark: Notice that if $P = M \times G$ is the trivial bundle then the equivariant maps $h : P \to G$ such that $h(p \cdot g) = g^{-1}h(p)g$ are completely determined by $h|_{M\times\{e\}}$. So for the trivial bundle the gauge group corresponds to the space $C^\infty(M, G)$. Note that we get $H(x, g) = H(x, e) \cdot g = (x, h(x)) \cdot g = (x, h(x)g)$.

Remark: This terminology differs from that of the physicist who call the fibre G the gauge group. They may refer to the gauge group described here the group of local symmetries.

Definition: (Gauge Group Action)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Consider the action of $\mathcal{G}_P$ on $\mathcal{A}_P$ defined such that for $K \in \mathcal{G}_P$ and $A \in \mathcal{A}_P$ we have the three following viewpoints of the action

- $H_{A\cdot K} = K^*(H_A)$
- $\varphi_{V_{A\cdot K}} = \varphi_{V_A} \circ TK$
- $\omega_{A\cdot K} = K^*(\omega_A)$

Lemma 1.1.3. *(Local Gauge Group Action)*
Let M be a smooth manifold and $(\pi : P = M \times G \to M, G, \cdot)$ the trivial be a principle bundle. Let $H \in \mathcal{G}_P$ and take $h : M \to G$ such that $H(x, g) = (x, h(x)g)$. We will abuse notation and take $h^{-1} : M \to G$ such that $h^{-1}(x) = (h(x))^{-1}$.

Let $A \in \mathcal{A}_P$ and $\omega_M \in \Omega^1(M, \mathfrak{g})$ the form associated to A using lemma C.2.3 such that for $(u, v) \in T_xM \times T_gG$ we have $(\omega_A)_{(x,g)}(u, v) = Ad_{g^{-1}}((\omega_M)_x(u)) + (\omega_G)_g(v)$ where ω_G is the Maurer-Cartan form. For $(u, v) \in T_xM \times T_gG$ we have

$$(H^*(\omega_A))_{(x,g)}(u, v) = Ad_{g^{-1}}(Ad_{h(x)^{-1}}((\omega_M)_x(u)) + (h^*(\omega_G))_x(u)) + (\omega_G)_g(v)$$

What this means is that for the trivial bundle $M \times G$ the action of the gauge group $C^\infty(M, G)$ acts on the space of connections viewed using lemma C.2.3 as $\Omega^1(M, \mathfrak{g})$ as follows

$$\alpha \cdot h = Ad_{h^{-1}}(\alpha) + h^*(\omega_G)$$

This is commonly found in the literature as the following expression which holds for matrix groups but notationally is slightly incorrect

$$\alpha \cdot h = h^{-1}\alpha h + h^{-1}dh$$

Proof. We have $H(x, g) = (x, h(x)g)$ and so $T_{(x,g)}H(u, v) = (u, T_{h(x)}R_g \circ T_xh(u) + T_gL_{h(x)}(v))$ as the differential of group multiplication at the identity is addition in the Lie algebra. So

$$\begin{aligned}
(H^*(\omega_A))_{(x,g)}(u, v) &= (\omega_A)_{x,h(x)g}(T_{(x,g)}H(u, v)) \\
&= (\omega_A)_{x,h(x)g}(u, T_{h(x)}R_g \circ T_xh(u) + T_gL_{h(x)}(v)) \\
&= Ad_{(h(x)g)^{-1}}((\omega_M)_x(u)) + (\omega_G)_{h(x)g}(T_{h(x)}R_g \circ T_xh(u) + T_gL_{h(x)}(v)) \\
&= Ad_{g^{-1}}(Ad_{h(x)^{-1}}((\omega_M)_x(u))) + (\omega_G)_{h(x)g}(T_{h(x)}R_g \circ T_xh(u)) + (\omega_G)_{h(x)g}(T_gL_{h(x)}(v)) \\
&= Ad_{g^{-1}}(Ad_{h(x)^{-1}}((\omega_M)_x(u))) + Ad_{g^{-1}}((\omega_G)_{h(x)g}(T_xh(u))) + (\omega_G)_g(v) \\
&= Ad_{g^{-1}}(Ad_{h(x)^{-1}}((\omega_M)_x(u)) + (h^*(\omega_G))_x(u)) + (\omega_G)_g(v)
\end{aligned}$$

□

Remark: This lemma is important as it describes the action of the gauge group locally as locally all bundles are trivial.

Lemma 1.1.4. *(Gauge Group and Curvature)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle, $H \in \mathcal{G}_P$ and $A \in \mathcal{A}_P$. Then if $F_A \in \Omega^2(P, \mathfrak{g}_P)$ is the curvature of A we have

$$F_{A \cdot H} = H^*(F_A)$$

Proof.

$$\begin{aligned}
F_{A \cdot H} = d\omega_{A \cdot H} + [\omega_{A \cdot H} \wedge \omega_{A \cdot H}] &= d(H^*\omega_A) + [H^*(\omega_A) \wedge H^*(\omega_A)] \\
&= H^*(d\omega_A + [\omega_A \wedge \omega_A]) = H^*(F_A)
\end{aligned}$$

□

Lemma 1.1.5. *(Gauge Group Action on $T\mathcal{A}_P$)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Consider the action of $\mathcal{G}_P$ on $\mathcal{A}_P$. There is a natural action of $\mathcal{G}_P$ on $T\mathcal{A}_P$ defined for $\alpha_A \in T_A\mathcal{A}_P$ and $H \in \mathcal{G}_P$ such that $\alpha_A \cdot H = T_A R_H(\alpha_A)$. If $h \in C^\infty(P, G)$ is the equivariant map associated to H then

$$\alpha_A \cdot H = Ad_{h^{-1}}(\alpha_A) \in \Omega^1(M, \mathfrak{g}_P) = T_{A\cdot H}\mathcal{A}_P$$

where for $\alpha_x(X_x) = (p, v)$ we have $Ad_{h^{-1}}(\alpha_A)_x(X_x) = (p, Ad_{h(p)^{-1}}(v)) = (p\cdot h(p)^{-1}, v)$ which is well defined as h is equivariant.

Proof. For $A_t = A + t\alpha_A \in \mathcal{A}_P$ we have $\alpha_A = \lim_{t\to 0} \frac{A_t - A}{t}$. So $\alpha_A \cdot H = \lim_{t\to 0} \frac{A_t\cdot H - A\cdot H}{t}$. This then means that locally we have

$$\alpha_A \cdot H = \lim_{t\to 0} \frac{A_t \cdot H - A \cdot H}{t} = \lim_{t\to 0} \frac{Ad_{h^{-1}}(\omega_{A_t}) + h^*(\omega_G) - Ad_{h^{-1}}(\omega_A) - h^*(\omega_G)}{t}$$
$$= Ad_{h^{-1}} \lim_{t\to 0} \frac{\omega_{A_t} - \omega_A}{t} = Ad_{h^{-1}}(\alpha_A)$$

So locally the action is given by $Ad_{h^{-1}}(\alpha_A)$. This doesn't depend on any local data and so can be seen to extend to the global form α_A. □

> **Remark:** As $\mathcal{A}_P$ is affine $T\mathcal{A}_P \cong \mathcal{A}_P \times \Omega^1(M, \mathfrak{g}_P)$ and this argument holds. However to make sense of $T\mathcal{A}_P$ in general requires some work as $\mathcal{A}_P$ is infinite dimensional.

Gauge Group as an Infinite dimensional Lie Group: We can think of the gauge group as an infinite dimensional Lie group. Noting the gauge group is given by $\Gamma(Ad(P))$ we can identify the "Lie algebra" of the gauge group as $T_e\Gamma(Ad(P)) = \Gamma(ad(P)) = \Omega^0(M, \mathfrak{g}_P)$.

For $h \in \Gamma(Ad(P))$, $\alpha \in \Gamma(ad(P))$, $x \in M$ and $(p, v) = \alpha(x)$ we have an analogy of the adjoint action given by $Ad_h(\alpha)(x) = (p, Ad_{h(x)}v)$. We have the following analogy of the exponential map $\exp(\alpha)(x) = (p, exp(v))$. A Lie bracket for $\alpha, \beta \in \Omega^0(M, \mathfrak{g}_P)$ is given by $[\alpha \wedge \beta] \in \Omega^0(M, \mathfrak{g}_P)$.

There is an analogy of the fundamental vector field associated to an element of the Lie algebra $\alpha \in \Omega^0(M, \mathfrak{g}_P)$. This is an element $X \in \mathfrak{X}(\mathcal{A}_P)$ such that $X_A = T_A L_A(\beta)$.

Lemma 1.1.6. *(Infinitesimal Gauge Group Action)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. For $\beta \in T_e\mathcal{G}_P = \Omega^0(M, \mathfrak{g}_P)$ and $A \in \mathcal{A}_P$ we have

$$T_A L_A(\beta) = \frac{d}{dt} A \cdot \exp(t\beta) = d_A\beta$$

Where d_A is the covariant derivative. See Appendix D.

Proof. If A is given locally by α then from lemma 1.1.3 locally we have

$$\frac{d}{dt} A \cdot \exp(t\beta) = \frac{d}{dt}(\exp(t\beta)\alpha \exp(-t\beta) + \exp(-t\beta) d \exp(t\beta))$$
$$= \frac{d}{dt}(\alpha + t(\beta\alpha - \alpha\beta) + ... + td\beta + ...) = 2[\beta \wedge \alpha] + d\beta = d_A\beta$$

where the last equality comes from lemma D.2.3. □

Remark: Viewing the gauge group as an infinite dimensional manifold can be made rigorous, however, is out of the scope and purpose of this thesis. For those interested in some of the foundations of infinite dimensional manifolds consult [KM97]. There is then a discussion of the spaces involved in section 14 of [AB83].

1.1.3 The Moduli Space of Flat Connections

Definition: (Flat Connections)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. If for each point $p \in P$ if there exists an open $U \subseteq M$ of $\pi(p)$ such that $(\pi^{-1}(U), A|_{\pi^{-1}(U)}) \cong (U \times G, TM \times \{0\})$ we say that A is **flat**.

That is, a connection is flat if the connection is given locally by the trivial connection. Denote the set of flat connections on P by $\mathcal{A}_{P,flat}$.

Lemma 1.1.7. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. $A \in \mathcal{A}_P$ is flat if and only if $F_A = 0$.*

Proof. Using lemma D.2.4 and Frobenius's Theorem A.3.2 the result follows. As from lemma D.2.4 we can see that $H_A \subseteq TP$ is an integrable sub-bundle of TP, that is it is closed under the Lie bracket, and Frobenius's theorem says that this is true if and only if H_A is defined by a foliation, that is locally the connection is given by the trivial connection. □

Definition: (Moduli Space of Flat Connections on a principle bundle P, $\mathcal{R}_P$)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $\mathcal{A}_{P,flat} \subseteq \mathcal{A}_P$ be the set of flat connections with subspace topology. Define $\mathcal{R}_P = \mathcal{A}_{P,flat}/\mathcal{G}_P$ to be the **moduli space of flat connections on** P.

Definition: (Moduli Space of Flat G-connections on M, $\mathcal{R}_{M,G}$)
Let M be a smooth manifold. Let

$$\mathcal{P}_G = \{\text{Isomorphism classes of prinicple } G\text{-bundles over } M\}$$

and define the **moduli space of flat connections over** M to be given by

$$\mathcal{R}_{M,G} = \bigsqcup_{P \in \mathcal{P}_G} \mathcal{R}_P$$

Remark: This is well defined as the gauge group is a special class of automorphisms of P and so it sends integrable sub-bundles of the tangent bundle to integrable sub-bundles. That is, the gauge group sends flat connections to flat connections. Also from lemma 1.1.4 we see that $F_{A \cdot H} = H^* F_A = H^*(0) = 0$.

Remark: The moduli space $\mathcal{R}_{M,G}$ can be given more structure in various cases. For surfaces, the theory is well documented [AB83] [Aud04] [Gol86] [Tyu03] and $\mathcal{R}_{M,G}$ can be given the structure of a complex manifold, Kähler manifold or symplectic manifold on a dense set of $\mathcal{R}_{M,G}$ with a few singularities.

1.1.4 The Representation Variety and Character Variety

The moduli space arises from the differential geometry of connections. There are some closely related spaces coming from algebra that give rise to the same space.

Definition: (Representation Variety)
Let G be a Lie group and Γ be a group. We define the **representation variety** of Γ with respect to G as

$$Hom(\Gamma, G)$$

with the topology inherited by G.

Definition: (Character Variety)
Let G be a Lie group and Γ be a group. We define the **character variety** of Γ with respect to G as

$$Hom(\Gamma, G)/G$$

where G acts on $Hom(\Gamma, G)$ by point-wise conjugation and the topology is inherited by G. That is for $\rho \in Hom(\Gamma, G)$, $a \in \Gamma$ and $g \in G$ we have $(\rho \cdot g)(a) = g^{-1}\rho(a)g$.

Theorem 1.1.8. *Let M be a path connected smooth manifold, G a Lie group and $x \in M$. The holonomy map (of corollary E.2.2) gives a bijection*

$$\mathcal{R}_{M,G} \cong Hom(\pi_1(M, x), G)/G$$

Proof. **Injectivity:** Let $(\pi_P : P \to M, G, \cdot)$ and $(\pi_Q : Q \to M, G, \cdot)$ be principle bundles, $A \in \mathcal{A}_{P,flat}$ and $B \in \mathcal{A}_{Q,flat}$ with $\rho_A, \rho_B \in Hom(\pi_1(M, x), G)/G$ associated to A and B by corollary E.2.2 such that $\rho_A = \rho_B$.

We wish to show that there exists an equivariant bundle map that covers the identity $H : P \to Q$ such that $A = H^*(B)$. This will show that $P \cong Q$ and with this association $A = B$.

Notice that as $\rho_A = \rho_B$ there exists $p \in \pi_P^{-1}(x)$ and $q \in \pi_Q^{-1}(x)$ such that $\rho_{A,p} = \rho_{B,q}$ where $\rho_{A,p}$ is the holonomy representation of A based at p and $\rho_{B,q}$ is the holonomy representation of B based at q (simply adjusting $q \mapsto q \cdot g$).

Consider $y \in M$ and let $\gamma : [0, 1] \to M$ be a piecewise smooth path in M such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $\widetilde{\gamma}_A : [0, 1] \to P$ be the horizontal lift of γ with respect to A and the point p and $\widetilde{\gamma}_B : [0, 1] \to Q$ be the horizontal lift of γ with respect to B and the point q.

Define $H : P \to Q$ to be the equivariant bundle map such that $H(\widetilde{\gamma}_A(1)) = \widetilde{\gamma}_B(1)$. Note that this determines the function H completely as M is path connected and H is defined to be equivariant. We want to show that this is well defined and independent of the path γ and the points p and q.

It can be shown that H is smooth by noting that it is defined as the solution of a differential equation and by its definition it is a bundle map. For $\delta : [0,1] \to M$ another path such that $\delta(0) = x$ and $\delta(1) = y$ and let $\widetilde{\delta}_A : [0,1] \to P$ be the horizontal lift of δ with respect to A and the point p and $\widetilde{\delta}_B : [0,1] \to Q$ be the horizontal lift of δ with respect to B and the point q. We have the following where we let $\overline{\gamma}$ be the path flowing in the opposite direction and $\gamma \star \delta$ the path of γ followed by δ.

$$H(\widetilde{\gamma}_A(1) \cdot \rho_{A,p}(\overline{\delta} \star \gamma)) = H(\widetilde{\gamma}_A(1) \cdot \rho_{A,\widetilde{\gamma}_A(1)}(\overline{\gamma} \star \delta)) = H(\widetilde{\delta}_A(1))$$

$$= \widetilde{\delta}_B(1) = \widetilde{\gamma}_B(1) \cdot \rho_{B,\widetilde{\gamma}_B(1)}(\overline{\gamma} \star \delta) = \widetilde{\gamma}_B(1) \cdot \rho_{B,q}(\overline{\delta} \star \gamma)$$

Notice that $\rho_{A,p}(\overline{\delta} \star \gamma) = \rho_{B,q}(\overline{\delta} \star \gamma)$. Therefore if two paths reach the same end point the loop they form has trivial holonomy in both P and Q and so H is well defined.

Now H was also defined to be equivariant which by the previous computation is also shown to be well defined on all of P. Therefore we see that A and B are gauge equivalent flat connections as from the definition of H we have $H^*(B) = A$ as horizontal vectors are mapped to horizontal vectors.

Surjectivity: Let $\rho \in Hom(\pi_1(M,x), G)$. Let $\pi : \widetilde{M} \to M$ be the universal cover of M. Notice that $\pi_1(M,x)$ acts on $\widetilde{M}$ via its association with the Deck transformations of $\pi : \widetilde{M} \to M$.

Let $P = \widetilde{M} \times_{\pi_1(M,x)} G$ where $\widetilde{M} \times_{\pi_1(M,x)} G = \widetilde{M} \times G / \sim$ with $(\widetilde{y} \cdot [\gamma], g) \sim (\widetilde{y}, \rho([\gamma])g)$. Notice we have $P = \widetilde{M} \times G/\pi_1(M,x)$ such that $\pi_1(M,x)$ acts on $\widetilde{M} \times G$ such that $(\widetilde{x}, g) \cdot [\gamma] = (\widetilde{x} \cdot [\gamma], \rho([\gamma])^{-1} g)$. Define $\pi_P : P \to M$ to be $\pi_P[\widetilde{y}, g] = \pi(\widetilde{y})$. This is well defined as the action of $\pi_1(M,x)$ on $\widetilde{M}$ fixes the fibres of π. Define the action of G on P such that for $g' \in G$ and $[\widetilde{y}, g] \in P$ we have $[\widetilde{y}, g] \cdot g' = [\widetilde{y}, gg']$.

We can see that $(\pi_P : P \to M, G, \cdot)$ defines a principle fibre bundle. Notice that $TP = T\widetilde{M} \times_{\pi_1(M,x)} TG = T\widetilde{M} \times TG/\pi_1(M,x)$ where the action of $\pi_1(M,x)$ on $\widetilde{M}$ and G lifts to an action on $T\widetilde{M}$ and TG. Let $A \in \mathcal{A}_P$ such that $\varphi_{H_A} = \{[v, 0] \in T\widetilde{M} \times TG/\pi_1(M)\}$. This is well defined and is flat as it is closed under the Lie bracket which is a property that descends to the quotient.

Now let $[\gamma] \in \pi_1(M,x)$. Consider $\widetilde{\gamma}$ the lift of γ to the universal cover with $\widetilde{\gamma}(0) = \widetilde{x} \in \pi^{-1}(x)$. Then γ lifts to the path $[\widetilde{\gamma}, e]$ in P. Notice that $[\widetilde{\gamma}, e](1) = [\widetilde{x} \cdot [\gamma], e] = [\widetilde{x}, \rho([\gamma])] = [\widetilde{\gamma}, e](0) \cdot \rho([\gamma])$.

This shows that P with A is a principle G bundle with flat connection such that the holonomy representation class is given by $[\rho]$. □

Remark: This association gives the moduli space a topology.

Remark: For smooth manifolds the fundamental group is finitely generated and therefore the character variety $Hom(\pi_1(M,x), G)/G$ is a finite dimensional space. This shows that the moduli space of flat connections is in fact finite dimensional.

Remark: This theorem says that a gauge equivalence class of flat connections determines G-covers of our manifold where G is taken to have the discrete topology. Note that these covers will locally be homeomorphic to $\mathbb{R}^n$ however in general their topologies will have similar properties of the so-called "long line".

Lemma 1.1.9. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle with M path connected. Let $A \in \mathcal{A}_{P,flat}$. Let $\rho_A \in Hom(\pi_1(M,x), G)$ be a representative of the representation of the fundamental group determined by A. Then we have $Stab_{\mathcal{G}_P}(A) \cong Stab_G(\rho_A)$.*

Proof. Suppose that $h : P \to G$ such that $h(p \cdot g) = g^{-1}h(p)g$. Suppose that $H : P \to P$ such that $H(p) = p \cdot h(p)$ and that $A \cdot H = A$.

As $A \cdot H = A$ for any horizontal path in (P, A) say $\tilde{\gamma} : [0,1] \to P$ with $\tilde{\gamma}(0) = p$ we have $H \circ \tilde{\gamma} \cdot h(p)^{-1} = \tilde{\gamma}$ as $H \circ \tilde{\gamma} \cdot h(p)^{-1}$ and $\tilde{\gamma}$ are both horizontal lifts of $\pi(\tilde{\gamma})$ with

$$H \circ \tilde{\gamma} \cdot h(p)^{-1}(0) = H(p) \cdot h(p)^{-1} = p \cdot h(p)h^{-1}(p) = p = \tilde{\gamma}(0)$$

So $H \circ \tilde{\gamma} = \tilde{\gamma} \cdot h(p)$. Notice that as M is path connected for $y \in M$ we have $\gamma : [0,1] \to M$ such that $\gamma(0) = \pi(p)$ and $\gamma(1) = y$. So $\tilde{\gamma} : [0,1] \to P$ is a horizontal lift such that $\tilde{\gamma}(0) = p$ and take $q = \tilde{\gamma}(1) \in \pi^{-1}(y)$. Then $H \circ \tilde{\gamma}(1) = H(q) = \tilde{\gamma}(1) \cdot h(p)$.

Notice that as H is equivariant H is determined by one element on each fibre. As we have $H(q) = \tilde{\gamma}(1) \cdot h(p)$ for $\tilde{\gamma}(1) = q \in \pi^{-1}(x)$. We can see that H is determined completely by $h(p)$ and A as it determines the horizontal lifts $\tilde{\gamma}$.

Recall that for $[\gamma] \in \pi_1(M, x)$ we have $\tilde{\gamma} : [0,1] \to P$ with $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) = p \cdot \rho_A([\gamma])$. So we have

$$p \cdot h(p)\rho_A([\gamma]) = H(p) \cdot \rho_A([\gamma]) = H(p \cdot \rho_A([\gamma])) = H \circ \tilde{\gamma}(1) = \tilde{\gamma}(1) \cdot h(p) = p \cdot \rho_A([\gamma])h(p)$$

This means that $h(p)\rho_A([\gamma]) = \rho_A([\gamma])h(p)$ and so $h(p)^{-1}\rho_A([\gamma])f(p) = \rho_A([\gamma])$. We can identify $H \mapsto h(p)$ and note that for all $[\gamma] \in \pi_1(M, x)$ we have

$$h(p)^{-1}\rho_A([\gamma])h(p) = \rho_A([\gamma])$$

Therefore $h(p) \in Stab_G(\rho_A)$. We have $Stab_{\mathcal{G}_P}(A) \leqslant Stab_G(\rho_A)$.

Notice that for $g \in Stab_G(\rho_A)$ we can define $g \mapsto H$ such that $h(p) = g$. This means that $Stab_{\mathcal{G}_P}(A) \geqslant Stab_G(\rho_A)$. This completes the proof. □

Remark: This is important as it says that the failure of the action of the gauge group $\mathcal{G}_P$ to be free on $A \in \mathcal{A}_P$ is measured by the failure of the action of G on $\rho_A \in Hom(\pi_1(\Sigma, x), G)$ to be free. From the outset quantifying the second is a much easier task. Moreover this says that the singularities induced from the orbits that aren't free are given by the same points in both viewpoints of the moduli space.

1.2 The Moduli Space of Flat Connections on Circles and Surfaces

The following section will describe the moduli space in dimensions 1 and 2. For a 0-dimensional base everything is trivial. The next case is in dimension 1. This example can be understood using theorem 1.1.8. Alternatively, it can be understood using gluing techniques. The moduli space of flat connections over surfaces can also be understood using theorem 1.1.8 and gluing techniques. The complexity involved in the topology of moduli space over surfaces already starts to increase.

1.2.1 The Moduli Space of Flat Connections on the Circle

The circle is a very concrete example and is in fact the only connected compact 1-dimensional smooth manifold without boundary so is in this sense important. We have the following result.

Theorem 1.2.1.

$$\mathcal{R}_{S^1,G} = G/Ad(G)$$

Proof. (Using theorem 1.1.8)
$\pi_1(S^1) = \mathbb{Z}$ and so $\mathcal{R}_{S^1,G} = Hom(\pi_1(S^1)G)/G = Hom(\mathbb{Z}, G)/G = G/Ad(G)$. □

Now we will elucidate some hidden aspects of this example that are not from the outset clear from the proof of theorem 1.1.8.

Lemma 1.2.2. *Let $(\pi : P \to [0,1], G, \cdot)$ be a principle bundle. Then P is trivial.*

Proof. We can find $a_0, ..., a_n \in [0,1]$ such that $\bigcup_{k=0}^{n-1}[a_k, a_{k+1}] = [0,1]$ on which $\pi^{-1}([a_k, a_{k+1}])$ is trivial. Therefore given an initial point and tangent vector $\pi^{-1}(a_k)$ we can define a section of $\pi^{-1}([a_k, a_{k+1}])$ with initial point and tangent vector. We can then inductively build up a section on P by using the local sections that agree on $\pi^{-1}(a_k)$. □

Lemma 1.2.3. *Let $(\pi : [0,1] \times G \to [0,1], G, \cdot)$ be the trivial bundle. Then every connection on $[0,1] \times G$ is gauge equivalent to the trivial connection.*

Proof. Each connection defines and is defined by some smooth section of $[0,1] \times G$ or path in G but we can the take this path composed with the inverse in G to get a gauge transformation that trivialises the connection. □

Lemma 1.2.4. *Let $S^1 = [0,1]/\sim$ with $0 \sim 1$ and $(\pi : P \to S^1, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let $(\pi' : P' \to [0,1], G, \cdot)$ be the principle bundle induced on $[0,1]$ by S^1 and A' the connection induced from A. Then by the lemma 1.2.2 there exists $\varphi : [0,1] \times G \cong P'$. Then there exists $H \in \mathcal{G}_P$ such that $\varphi^*(A \cdot H)$ has $(H_{\varphi^*(A \cdot H)})_{0,g} = \{(v,0) \in T_0[0,1] \times T_gG\}$ and $(H_{\varphi^*(A \cdot H)})_1 = \{(v,0) \in T_1[0,1] \times T_gG\}$.*

Proof. We use a bump function and the exponential map of G to construct a smooth function $C^\infty([0,1], G)$ that is associated to H. □

Combining these result we can now make use of the fact that $S^1 = [0,1]/\sim$ where $0 \sim 1$ to get some understanding of the structure of $\mathcal{R}_{S^1,G}$ by glueing principle bundles with connections on $[0,1]$.

Corollary 1.2.5. *Let $S^1 = [0,1]/\sim$ with $0 \sim 1$ and $(\pi : P \to S^1, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Then there exists $g \in \mathcal{G}_{G\times point} = G$ such that for the trivial section on $[0,1] \times G$ we have $(P, A) \cong ([0,1] \times G, T)/\sim$ where $(1, h) \sim (0, gh)$ where T then induces a connection on $[0,1] \times G/\sim$ as $(H_T)_0 = (H_T)_1$.*

Proof. From the last lemmas we can "flatten" P at $0 \sim 1$ and then noting that this is defined by gluing a connection on $[0,1]$ we then see that this connection is then gauge equivalent to the trivial connection. The gauge equivalence must fix the space at 0 and 1 and so we see that all along $(\pi : P \to S^1, G, \cdot, A)$ was defined by some gluing of the trivial bundle on T with the trivial connection. □

This shows that gluing together gauge equivalence classes of flat connections on $[0,1]$ makes sense and in fact gives rise to all possible gauge equivalence classes on S^1. To classify gauge equivalence classes of flat connections on S^1 is reduced to understand the gluing of $[0,1]$ bundles and the gauge equivalence classes of flat connections on $[0,1]$. Notice that we showed the latter to be trivial and that the original was given by G. The next theorem determines when different gluing determine gauge equivalence connections.

Theorem 1.2.6.

$$\mathcal{R}_{S^1,G} = G/Ad(G)$$

Proof. There is only one gauge equivalence class of flat connections on $[0,1]$. To determine the gauge equivalence classes of flat connections on S^1 by the previous lemmas and corollary, we must determine what flat connections induced by glueing this trivial connection are gauge equivalent. Considering the gluing we have $(0, gh) \sim (1, h)$ and $(0, jh) \sim (1, h)$ induce the same gauge equivalence class of flat connections if and only if $j = k^{-1}gk$. This is because by choosing the trivial representative of the flat connection we haven't completely used all possible gauge transformations. We can take the gauge transformations given by $(x, g) \mapsto (x, kg)$ for some $k \in G$. This means that we have $(0, gh) \mapsto (0, kgh)$ and $(1, h) \mapsto (1, kh)$. In particular under this gauge transformation we have $(0, kgh) \sim (1, kh)$ and so $(0, kgk^{-1}kh) \sim (1, kh)$. In other words $(0, kgk^{-1}h) \sim (1, h)$. This shows that all gauge equivalence classes of flat connections on S^1 are given by elements of $G/Ad(G)$. □

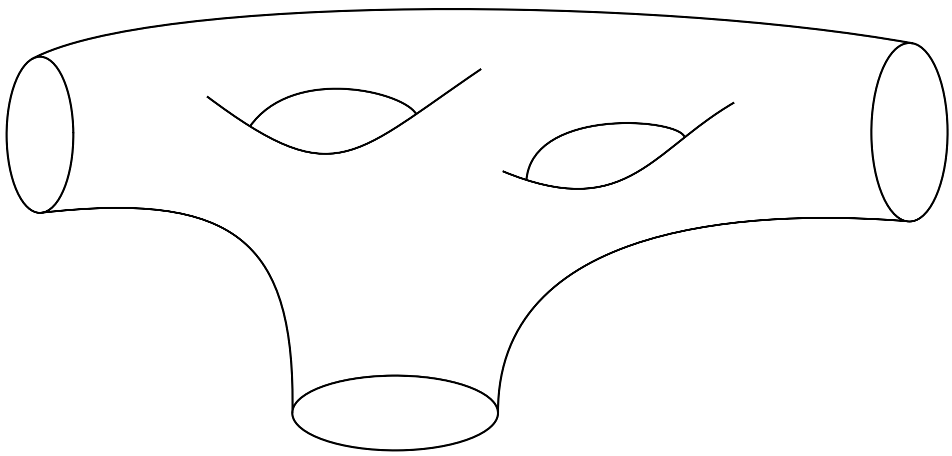
This example illustrates the importance of gluing. Connections are defined by local information and the gauge equivalence is also defined by local information. Gluing together spaces with connections therefore makes sense in the same way that manifolds are locally defined objects and we can glue them together. Understanding some basic examples allows us to understand the global picture by gluing together the simpler pieces. This then reduces the problem to understanding the gluing.

1.2.2 Trinion Decompositions and the Pair of Pants

The moduli space of flat connections on a surface has been extensively studied. See [AB83] [Aud04] [Gol84] [Gol86] [JW92] [JW94] [Wit91].

There is a classification of surfaces with boundary and their fundamental groups are known. This allows an explicit description of the moduli space of flat connections on a surface. Gluing then allows further understanding as shown in section 1.2.1.

Notation: From the classification of surfaces we know that compact orientable (note: not oriented) surfaces with boundary are classified by their genus g and the number of boundary circles n. For example the surface with three boundary circles of genus two is depicted here.



So we denote the compact orientable surface of genus g with n boundary circles as $\Sigma_{g,n}$. The surface above is then denoted by $\Sigma_{2,3}$. We will denote the moduli space of flat connections on $\Sigma_{g,n}$ with fibre G as $\mathcal{R}_{G,g,n}$.

> **Remark:** We have the following standard presentation for the fundamental group of $\Sigma_{g,n}$.
>
> $$\pi_1(\Sigma_{g,n}, x) = \left\langle a_1, b_1, ..., a_g, b_g, c_1, ..., c_n | a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} c_1 ... c_n \right\rangle$$
>
> See section 1 of [Hat02] for more on the fundamental group. Note that in particular for $n > 0$ we have $\pi_1(\Sigma_{g,n}, x) \cong F_{2g+n-1}$ the free group on $2g + n - 1$ generators. This means that for $n > 0$ the representation theory of this group is trivial.

Example: Let T be an Abelian Lie group. Consider

$$\mathcal{R}_{T,g,n} \cong Hom(\pi_1(\Sigma_{g,n}, x), T)/T$$

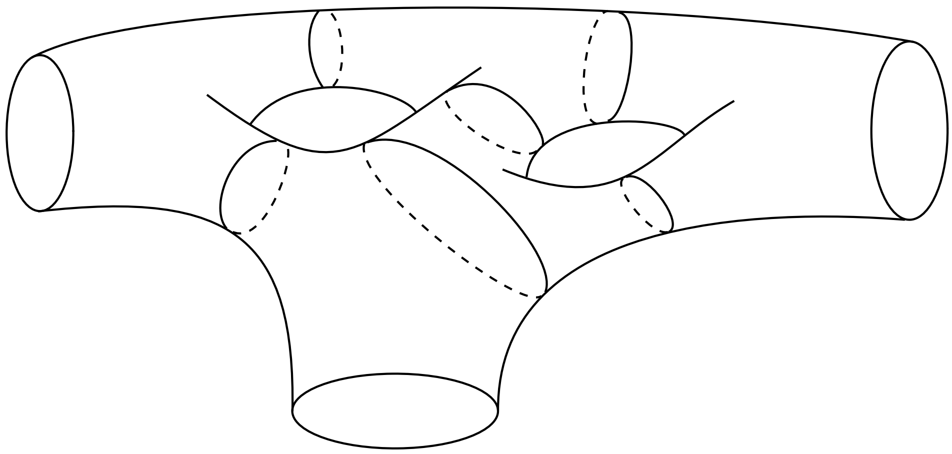
Noting that as T is Abelian conjugation is given by the trivial group action and we have $\mathcal{R}_{T,g,n} \cong Hom(\pi_1(\Sigma_{g,n}, x), T)$. We can then see from the previous remark that if $n \neq 0$ we have

$$\mathcal{R}_{T,g,n} \cong T^{2g-1+n}$$

and if $n = 0$ we have

$$\mathcal{R}_{T,g,0} \cong T^{2g}$$

Pair of Pants Decompositions: Every hyperbolic surface with boundary can be obtained by glueing together three-holed spheres along their boundaries. The three-holed sphere is often referred to the pair of pants or the trinion. Given a hyperbolic surface, there is no unique decomposition into pairs of pants. The number of pairs of pants in a decomposition is given by the negative of the Euler characteristic. An example is given here



This makes the pair of pants a very important surface when dealing with locally objects. It will play the role of the interval in the previous section 1.2.1. We have the following definitions and lemma.

Definition: (Trinion Decomposition)
Let Σ be a surface. Let Γ be a set of circles in Σ such that $\Sigma - \Gamma$ is a disjoint union of trinions $Trin$. We say that $Trin$ is a **trinion decomposition** with Γ the set of circles.

Remark: We are only really interested in trinion decompositions up to isotopy.

Definition: (Connection Adapted to a Trinion Decomposition)([JW92] Definition 2.2)
Let Σ be a compact orientable surface with or without boundary, $Trin$ a trinion decomposition with circles given by Γ and let $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. We say A is a **adapted to a trinion decomposition** of for each circle $\gamma \in \Gamma$ in the trinion decomposition

there is a tubular neighbourhood of the circle $\gamma \subseteq U$ with local coordinates $(x,\theta)^{-1} : (0,1) \times S^1 \cong U$ such that $((x,\theta)^{-1})^*(\omega_A|_U) = v d\theta$ for some $v \in \mathfrak{g}$.

Remark: This means that the connection looks like the canonical flat connection on the circle with holonomy given by $Hol([\gamma])$ for each $\gamma \in \Gamma$.

Lemma 1.2.7. *([Wit91] section 4.5)*
Let G a semi-simple Lie group and let $T \leqslant G$ be the maximal torus. Let $Trin$ be a trinion decomposition of $\Sigma_{g,n}$. Then the map

$$g_{Trin} : \mathcal{R}_{G,g,n} \to \left\{([A_t])_{t \in Trin} \in (\mathcal{R}_{G,0,3})^{Trin} : \textit{for } t_1, t_2 \in Trin \textit{ with } \gamma \in \partial t_1 \textit{ and } \gamma \in \partial t_2 \textit{ we have } [(A_{t_1})|_\gamma] = [(A_{t_2})|_\gamma]\right\} \quad s.t \quad g_{Trin}(A) = (A|_t)_{t \in Trin}$$

is surjective with generic fibres (i.e pre-images of generic points are) given by $T^\Gamma/\mathcal{L}$ where $\mathcal{L} \leqslant T^\Gamma$ is a subgroup of order $\#Z(G)^{2g-3}$ where $Z(G)$ is the center of G.

Proof. Given each $\gamma \in \Gamma$, $t_1, t_2 \in Trin$ with $\gamma \in \partial t_1$ and $\gamma \in \partial t_2$ and for a representative of $[A_t] \in \mathcal{R}_{G,0,3}$ say $A_{t_i} \in \mathcal{A}_{\pi_{t_i}: P_{t_i} \to \Sigma_{0,3}, flat}$ we have a gauge equivalence $\Phi_\gamma : P_{t_1} \to P_{t_2}$. We can construct a bundle on $\pi : P \to \Sigma_{g,n}$ by using Φ_γ to identify or glue the bundles $P_t \to \Sigma_{0,3}$ along each $\gamma \in \Gamma$. See pg. 196 in [Wit91] for a brief discussion on the fibres but it can be shown generically that for $t \in T$ we have $t_\gamma \Phi_\gamma$ defines a set of gauge transformations that lead to the same gauge equivalence class of flat connections as the set of gauge transformations Φ_γ which shows that generically the fibre of the map is given by T^{3g-3+n}. ☐

Corollary 1.2.8. *(Gauge Equivalence Classes and Connections Adapted to a Trinion Decomposition)([JW92] lemma 2.3 and [Wit91] section 4.5)*
Let Σ be a compact orientable surface with or without boundary, $Trin$ a trinion decomposition with circles given by Γ and let $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. There exists $H \in \mathcal{G}_P$ such that $A \cdot H$ is adapted to the trinion decomposition $Trin$.

The space $\mathcal{R}_{G,0,3}$ can be used to build up information about the surfaces with smaller Euler characteristic. $SU(2)$ is the simplest non-abelian Lie group. The following lemma will be important later on.

Lemma 1.2.9. *($\mathcal{R}_{SU(2),0,3}$) [JW92] [Aud04]*

$$\begin{aligned}\mathcal{R}_{SU(2),0,3} &\cong \{(x,y,z) \in \mathbb{R}^3 : x \leqslant y+z,\ y \leqslant x+z,\ z \leqslant x+y,\ x+y+z \leqslant 2\} \\ &= \{(x,y,z) \in \mathbb{R}^3 : |x-y| \leqslant z \leqslant \min(x+y, 2-x-y)\}\end{aligned}$$

Proof. From theorem 1.1.8 we have the following.

$$\mathcal{R}_{SU(2),0,3} \cong Hom(\pi_1(\Sigma_{0,3}), SU(2))/SU(2) \cong \{(A,B,C) \in SU(2) : ABC = 1\}/SU(2)$$

Let $Tr(A) = 2\cos(\pi\theta_1)$, $Tr(A) = 2\cos(\pi\theta_2)$ and $Tr(A) = 2\cos(\pi\theta_3)$ for $\theta_1, \theta_2, \theta_3 \in [0,1]$. Every element of $SU(2)$ can be diagonalised by conjugation and so up to conjugation we can take

$$A = \begin{bmatrix} e^{i\pi\theta_1} & 0 \\ 0 & e^{-i\pi\theta_1} \end{bmatrix}$$

We can now conjugate B by diagonal matrices. That is the matrices that leave A fixed by conjugation. This means that we can choose

$$B = \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}$$

for $z \in \mathbb{C}$ and $w \in [0, 1]$. Notice that $Re(z) = \cos(\pi\theta_2)$ and $Re(z)^2 + Im(z)^2 + w^2 = 1$. Therefore $Im(z)^2 + w^2 = \sin^2(\pi\theta_2)$. So $w + iIm(z) = \sin(\pi\theta_2)(\cos(\pi\beta) + i\sin(\pi\beta))$ for some $\beta \in [0, 1]$ and so $w = \sin(\pi\theta_2)\cos(\pi\beta)$ and $z = \cos(\pi\theta_2) + i\sin(\pi\theta_2)\sin(\pi\beta)$.

$$AB = \begin{bmatrix} e^{i\pi\theta_1} & 0 \\ 0 & e^{-i\pi\theta_1} \end{bmatrix} \begin{bmatrix} \cos(\pi\theta_2) + i\sin(\pi\theta_2)\sin(\pi\beta) & \sin(\pi\theta_2)\cos(\pi\beta) \\ \sin(\pi\theta_2)\cos(\pi\beta) & \cos(\pi\theta_2) - i\sin(\pi\theta_2)\sin(\pi\beta) \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\pi\theta_1}(\cos(\pi\theta_2) + i\sin(\pi\theta_2)\sin(\pi\beta)) & e^{i\pi\theta_1}\sin(\pi\theta_2)\cos(\pi\beta) \\ e^{-i\pi\theta_1}\sin(\pi\theta_2)\cos(\pi\beta) & e^{-i\pi\theta_1}(\cos(\pi\theta_2) - i\sin(\pi\theta_2)\sin(\pi\beta)) \end{bmatrix}$$

Then we can see that $\cos(\pi\theta_3) = \cos(\pi\theta_1)\cos(\pi\theta_2) - \sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\beta))$. So

$$\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\beta)) = \cos(\pi\theta_1)\cos(\pi\theta_2) - \cos(\pi\theta_3)$$

If $\theta_1 \neq 0$ and $\theta_2 \neq 0$ we have

$$\sin(\pi\beta) = \frac{\cos(\pi\theta_1)\cos(\pi\theta_2) - \cos(\pi\theta_3)}{\sin(\pi\theta_1)\sin(\pi\theta_2)}$$

$$= \frac{\cos(\pi(\theta_1 + \theta_2)) + \sin(\pi\theta_1)\sin(\pi\theta_2) - \cos(\pi\theta_3)}{\sin(\pi\theta_1)\sin(\pi\theta_2)} = 1 + \frac{\cos(\pi(\theta_1 + \theta_2)) - \cos(\pi\theta_3)}{\sin(\pi\theta_1)\sin(\pi\theta_2)}$$

Note that as $0 \leqslant \sin(\pi\theta_1)\sin(\pi\theta_2)$ that $\sin(\pi\beta) \leqslant 1$ if and only if $\cos(\pi(\theta_1 + \theta_2)) - \cos(\pi\theta_3) \leqslant 0$ and so $\cos(\pi(2 - \theta_1 - \theta_2)) = \cos(\pi(\theta_1 + \theta_2)) \leqslant \cos(\pi\theta_3)$. Notice that on $[0, 2]$ that cos is non-increasing. This means that $\theta_3 \leqslant \theta_1 + \theta_2$ and $\theta_3 \leqslant 2 - \theta_1 - \theta_2$ by the symmetries of A, B, C we get

$$\theta_3 \leqslant \theta_1 + \theta_2, \quad \theta_2 \leqslant \theta_1 + \theta_3, \quad \theta_1 \leqslant \theta_2 + \theta_3 \quad \text{and} \quad \theta_1 + \theta_2 + \theta_3 \leqslant 2$$

Also notice that $\theta_1, \theta_2, \theta_3$ determine $\sin(\beta)$ and therefore $\cos(\beta)$ up to sign but $w \in [0, 1]$ then fixes that. This means that

$$\theta : \mathcal{R}_{SU(2),0,3} \to [0, 1] \quad s.t \quad \theta([A, B, C]) = (\theta_1, \theta_2, \theta_3)$$

is a bijection. We can see that this is also continuous and the inverse map is also continuous. □

Using theorem 1.1.8 we can determine the moduli space in higher dimensions as well.

Lemma 1.2.10. *($\mathcal{R}_{(S^1)^n,G}$)*
Let G be a compact semi-simple Lie group with maximal torus T and let $N(T)$ be the normaliser of T. Then if we let $N(T)$ act by simultaneous conjugation on T^n then we find

$$\mathcal{R}_{(S^1)^n,G} \cong T^n/N(T)$$

Proof. From theorem 1.1.8 we have the following.

$$\mathcal{R}_{(S^1)^n,G} \cong Hom(\mathbb{Z}^n, G)/G \cong \{A_1, ..., A_n \in G : A_iA_j = A_jA_i\}/G$$

By conjugating we can take $A_i \in T$ however as $A_iA_j = A_jA_i$ we must have $A_i \in T$. Taking $A \in T$ is well defined up to conjugation by elements of $N(T)$. $\square$

> **Remark:** Notice that $T \leqslant N(T)$ acts trivially on T^n so we really get an action of the Weyl group $W = N(T)/Z(T) = N(T)/T$ on T^n. For $n = 1$ we get $T/W = G/Ad(G)$ as expected.
>
> **Remark:** In higher dimensions every finitely generated group arises as the fundamental group of a smooth manifold of that dimension. The distinguishing features between higher dimensions is then encoded in the gluing.

1.2.3 Remark on a Complex and Smooth Structure in the Algebraic Case

For a surface Σ and Lie group G there is a natural smooth structure we can put on a dense set of $\mathcal{R}_{\Sigma,G}$. Considering the presentation

$$\pi_1(\Sigma_{g,n+1}, x) = \langle a_1, b_1, ..., a_g, b_g, c_1, ..., c_n\rangle$$

we can see that

$$Hom(\pi_1(\Sigma_{g,n}, x), G) \cong G^{2g+n-1}$$

The set of points that aren't free are given by $Z(G)^{2g+n} \subseteq G^{2g+n}$ where $Z(G)$ is the center of G. Noting that $Z(G)$ is closed we see $G^{2g+n} - Z(G)^{2g+n}$ is a dense open set of a smooth manifold and is therefore a smooth manifold. We then see that $(G^{2g+n} - Z(G)^{2g+n})/G$ is a smooth manifold. Note that $(G^{2g+n} - Z(G)^{2g+n})/G \subseteq Hom(\pi_1(\Sigma_{g,n+1}, x), G)/G$ is dense.

If we take G as a complex algebraic group then we see that

$$Hom(\pi_1(\Sigma_{g,n}, x), G) \subseteq G^{2g+n} \subseteq \mathbb{C}^{dim_{\mathbb{C}}(G)(2g+n)}$$

is defined by polynomial equations. This means that we can give

$$Hom(\pi_1(\Sigma_{g,n}, x), G)$$

the structure of a complex variety. We can sometimes take an algebro-geometric quotient $Hom(\pi_1(\Sigma_{g,n}, x), G)//G$ which gives the moduli space the structure of a variety.

There is an algebro-geometric viewpoint of the moduli space for certain groups such as $SU(n)$ and $U(n)$. There is a correspondence to the moduli space of a certain class of bundles over a given Riemann surface. This gives the moduli space the structure of a variety that depends on the Riemann surface.

See [Gol84] or [Gol85] for more in the algebraic case. For the algebro-geometric viewpoint see the review [Tha95]. For the analytic case see section 2.1.2 or [AB83].

Chapter 2

A Symplectic Structure on $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$

2.1 Atiyah-Bott Symplectic Form and a Symplectic Form on the Moduli Space of Flat Connections

Historically connections haven't been the standard example of a classical mechanical system or, in other words, a symplectic manifold. The first example was the theory of electro-magnetism. However, to generalise electro-magnetism it became important to consider systems with, what physicists call, local symmetries. These are physical theories with symmetries given by the space of smooth functions from the space-time of the theory to a Lie group. This is the classical framework on which modern particle physics is based. The gauge group of a trivial principle bundle is identified with exactly this space. When studying quantum mechanics, it is often useful to have a classical system to then quantise. This is the motivation to look for a symplectic structure on the space of connections. Notice that this may seem slightly difficult as the space of connections is an infinite dimensional affine space; however the fact that it is an affine space means that constructing a form reduces to finding a non-degenerate anti-symmetric 2-form.

When we have a 2-dimensional space-time we can in fact construct a symplectic form on the space of connections. The moduli space of flat connections is then in fact a symplectic or Hamiltonian reduction of the space of connections. This removes the degrees of freedom of defined by the gauge group symmetries.

There are many references on this subject for the approach through the theory of connections consult chapter V of [Aud04] and [AB83] for the algebraic approach via representations of the fundamental group see [Gol84] [Gol86] [Gol85] [Kar92] for quantisation using a real polarisation see [JW92] for some aspects related to the symplectic volume and intersection numbers and see [JW94] [Wei98] [Wit91] [Wit92] for some relation to algebraic geometry see [Tha95] and for a general overview see [Tyu03].

2.1.1 Definition and Moment Map

See section 3 of [AB83] for another presentation of these definitions and results. Notice that we have not presented all the information on infinite dimensional manifolds and their smooth structures. This is mentioned in section 14 of [AB83].

Definition: (Natural Riemannian Metric on $\mathfrak{g}_P$)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and consider the adjoint bundle $ad(P) = \mathfrak{g}_P$. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$ (for example the killing form). We can use this to define a **natural Riemannian metric** on $\mathfrak{g}_P$ as follows

$$\eta_{P,\kappa} \in \Gamma(\mathfrak{g}_P^* \otimes \mathfrak{g}_P^*) \quad s.t \text{ for } (p, v), (q, u) \in \pi_P^{-1}(x)$$

$$\text{we have } (\eta_{P,\kappa})_x((p, v), (q, u)) = \kappa(v, u)$$

This is well defined as κ is Ad-invariant and a smooth section as κ is smooth as inner products are smooth on $\mathbb{R}$-vector spaces with standard smooth structure.

Definition: (Functions from $\Omega^k(M, \mathfrak{g}_P) \otimes \Omega^l(M, \mathfrak{g}_P)$ to $\Omega^{k+l}(M)$)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$. We can use $\eta_{P,\kappa}$ to define a function from $\Omega^k(M, \mathfrak{g}_P) \otimes \Omega^l(M, \mathfrak{g}_P)$ to $\Omega^{k+l}(M)$ as follows

$$(\ \wedge\ )_\kappa : \Omega^k(M, \mathfrak{g}_P) \otimes \Omega^l(M, \mathfrak{g}_P) \to \Omega^{k+l}(M) \quad s.t \quad (\alpha \wedge \beta)_\kappa(X_1, ..., X_{k+l})$$

$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} sign(\sigma)\eta_{P,\kappa}(\alpha(X_{\sigma(1)}, ..., X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)}))$$

Lemma 2.1.1. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let φ_{H_A} be the horizontal projection associated to A. Let $\alpha, \beta \in \Omega^*(M, \mathfrak{g}_P)$. Then*

$$d(\alpha \wedge \beta)_\kappa = (d_A\alpha \wedge \beta)_\kappa \pm (\alpha \wedge d_A\beta)_\kappa$$

Proof. The covariant derivative acting forms on M with values in the adjoint bundle is given by $d_A\alpha = d\alpha + [\omega_A \wedge \alpha]$. So we get

$$(d_A\alpha \wedge \beta)_\kappa \pm (\alpha \wedge d_A\beta)_\kappa = (d\alpha \wedge \beta)_\kappa \pm (\alpha \wedge d\beta)_\kappa + ([\alpha \wedge \omega_A] \wedge \beta)_\kappa \pm (\alpha \wedge [\omega_A \wedge \beta])_\kappa$$

Notice that the invariance of κ under the adjoint action means that $\kappa([u, v], w) = \kappa(u, [v, w])$ which means that

$$([\alpha \wedge \omega_A] \wedge \beta)_\kappa = \pm(\alpha \wedge [\omega_A \wedge \beta])_\kappa$$

Therefore

$$(d_A\alpha \wedge \beta)_\kappa \pm (\alpha \wedge d_A\beta)_\kappa = (d\alpha \wedge \beta)_\kappa \pm (\alpha \wedge d\beta)_\kappa = d(\alpha \wedge \beta)_\kappa$$

□

Definition: (Atiyah-Bott Metric)
Let (M, η) be a Riemannian manifold and $(\pi : P \to M, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$. Define the following bilinear form on $\Omega^1(M, \mathfrak{g}_P)$ called the **Atiyah-Bott metric**.

$$g^{AB} : \Omega^1(M, \mathfrak{g}_P) \otimes \Omega^1(M, \mathfrak{g}_P) \to \mathbb{R} \quad s.t \quad g^{AB}(\alpha, \beta) = \int_M (\alpha \wedge *\beta)_\kappa$$

Where $*$ is the Hodge star operator associated to η.

Topological invariants can't depend on η and the Hodge star operator $*$. In dimension 2 we are lucky and can define the following form without the need of any Riemannian metric.

Definition: (Topological Atiyah-Bott Form for 2 Dimensional Manifolds)
Let Σ be a 2-manifold and $(\pi : P \to \Sigma, G, \cdot)$ a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$. Consider the space of connections $\mathcal{A}_P$.

Recall from section C.3 that $\mathcal{A}_P$ is an affine space based on $\Omega^1(\Sigma, \mathfrak{g}_P)$. This means there is a canonical association for $A \in \mathcal{A}_P$ given by $T_A\mathcal{A}_P = \Omega^1(\Sigma, \mathfrak{g}_P)$ noting that the tangent space of any affine space is canonically isomorphic to that vector space the affine space is based on. Using this canonical isomorphism we can define the following form called the **Atiyah-Bott symplectic form**

$$\omega^{AB,P,\kappa} \in \Omega^2(\mathcal{A}_P) \quad s.t \text{ for } \alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g}_P) = T_A\mathcal{A}_P$$

$$\text{we have } \omega_A^{AB,P,\kappa}(\alpha, \beta) = \int_\Sigma (\alpha \wedge \beta)_\kappa$$

Remark: $\mathcal{A}_P$ is an infinite dimensional smooth-manifold. The constructions of the tangent space and cotangent space will go through in this case. Notice that the de Rham complex will in general be infinitely long in this case.

Lemma 2.1.2. *($(\mathcal{A}_P, \omega^{AB,P,\kappa})$ is an Infinite Dimensional Symplectic Manifold)*
Let Σ be a 2-dimensional manifold and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$. $(\mathcal{A}_P, \omega^{AB,P,\kappa})$ is a symplectic manifold.

Proof. $\omega_A^{AB,P,\kappa}$ is constant with respect to the canonical trivialisation of $T\mathcal{A}_P = \mathcal{A}_P \times \Omega^1(\Sigma, \mathfrak{g}_P)$. This means that $d\omega^{AB,P,\kappa} = 0$ and $\omega^{AB,P,\kappa}$ is closed. To see that $\omega^{AB,P,\kappa}$ is non-degenerate we take a form $\alpha \in \Omega^1(\Sigma, \mathfrak{g}) - \{0\} = T_A\mathcal{A}_P - \{0\}$ and consider the function $\omega_A^{AB,P,\kappa}(\alpha, -) : \Omega^1(\Sigma, \mathfrak{g}_P) \to \mathbb{R}$.

As $\alpha \neq 0$ there exists an $x \in \Sigma$ such that $\alpha_x \neq 0$. By continuity there exists an open set $x \in U \subseteq \Sigma$ such that for $u \in U$ we have $\alpha_u \neq 0$ and by potentially taking a smaller open set around x we can make $\pi^{-1}(U) \cong U \times G$ and we have local coordinates for U given by $(x_1, x_2) : \Omega \to U$ for some open set $\Omega \subseteq \mathbb{R}^2$. Using the trivialisation we can view $\alpha|_U \in \Omega^1(U, \mathfrak{g})$. In particular we have $\alpha|_U = \alpha_1 dx_1 + \alpha_2 dx_2$ for smooth functions $\alpha_1, \alpha_2 \in C^\infty(\Sigma, \mathfrak{g})$.

Consider $\alpha_1(x) \in \mathfrak{g}$. By non-degeneracy of κ we can find $v \in \mathfrak{g}$ such that $\kappa(\alpha_1(x), v) = 1$. Consider $(\alpha|_U \wedge v dx_2)_\kappa = \kappa(v, \alpha_1) dx_1 \wedge dx_2$. Considering the function $\kappa(v, \alpha_1) : U \to \mathbb{R}$ by continuity there exists an open set $x \in V \subseteq U$ such that $\kappa(v, \alpha_1)|_V : V \to \mathbb{R}_{>0}$. Taking a bump function $f : M \to \mathbb{R}$ with $supp(f) \subseteq V$ and $f(x) = 1$ we can see that $\int_M (\alpha \wedge v f dx_2)_\kappa = \int_V (\alpha \wedge v f dx_2)_\kappa$ is the integral of a positive smooth function on an open set and so $0 < \int_V (\alpha \wedge v f dx_2)_\kappa$. Therefore $\omega_A^{AB,P,\kappa}(\alpha, -)$ is not the zero map and therefore $\omega_A^{AB,P,\kappa}$ is non-degenerate. □

Lemma 2.1.3. *(Gauge Group Action is Symplectic)*
Let Σ be a 2-dimensional manifold and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$. $\mathcal{G}_P$ acts on $(\mathcal{A}_P, \omega^{AB,P,\kappa})$ symplectically.

Proof. Let $H \in \mathcal{G}_P$ be an automorphism $H : P \to P$ that covers the identity with associated equivariant map $h : P \to G$. Let $\alpha, \beta \in \mathfrak{X}(\mathcal{A}_P)$ and $A \in \mathcal{A}_P$. Then

$$H^*(\omega^{AB,P,\kappa})_A(\alpha_A, \beta_A) = \omega_{A \cdot H}^{AB,P,\kappa}(T_A H(\alpha_A), T_A H(\beta_A))$$

From lemma 1.1.5 in section 1.1.2 we have $T_A H(\alpha_A) = Ad_{h^{-1}}(\alpha_A)$ and $T_A H(\beta_A) = Ad_{h^{-1}}(\beta_A)$. Notice that using the canonical identification $T\mathcal{A}_P = \mathcal{A}_P \times \Omega^1(\Sigma, \mathfrak{g}_P)$ we can identify $\omega_A = \omega_{A \cdot H}$. We then get

$$\begin{aligned} H^*(\omega^{AB,P,\kappa})_A(\alpha_A, \beta_A) &= \omega_A^{AB,P,\kappa}(Ad_{h^{-1}}(\alpha_A), Ad_{h^{-1}}(\beta_A)) \\ &= \int_\Sigma (Ad_{h^{-1}}(\alpha_A) \wedge Ad_{h^{-1}}(\beta_A))_\kappa \end{aligned}$$

Notice that κ is invariant under the adjoint action and so $(Ad_{h^{-1}}(\alpha_A) \wedge Ad_{h^{-1}}(\beta_A))_\kappa = (\alpha_A \wedge \beta_A)_\kappa$. So finally we see that

$$\begin{aligned} H^*(\omega^{AB,P,\kappa})_A(\alpha_A, \beta_A) &= \int_\Sigma (Ad_{h^{-1}}(\alpha_A) \wedge Ad_{h^{-1}}(\beta_A))_\kappa \\ &= \int_\Sigma (\alpha_A \wedge \beta_A)_\kappa = \omega_A^{AB,P,\kappa}(\alpha_A, \beta_A) \end{aligned}$$

□

This action is Hamiltonian. In fact, the moment map associated to the action of the gauge group with respect to the Atiyah-Bott symplectic form for closed surfaces is given by the curvature and for surfaces with boundary it is the curvature and the restriction of the gauge equivalence class of flat connections to the boundary circles.

Lemma 2.1.4. *(Curvature as a Non-Equivariant Moment Map)*
Consider $\Sigma_{g,n}$ the surface of genus g with n boundary circles and let $(\pi : P \to \Sigma_{g,n}, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$.

Consider the identification of $\mathcal{A}_P \cong \Omega^1(M, \mathfrak{g}_P)$ given by $\varphi_B(A) = A - B$. The action of $\mathcal{G}_P$ on $(\mathcal{A}_P, \omega^{AB,P,\kappa})$ is Hamiltonian with moment map given by $\mu : \mathcal{A}_P \to \Omega^0(M, \mathfrak{g}_P)^*$ such that $\mu(A)(\alpha) = \int_{\Sigma_{g,n}} (\alpha \wedge F_A)_\kappa - \int_{\partial\Sigma_{g,n}} (\alpha \wedge \varphi_B(A))_\kappa$. Notice that μ depends on κ and B.

Proof. Consider $\mu^\alpha : \mathcal{A}_P \to \mathbb{R}$ such that $\mu^\alpha(A) = \mu(A)(\alpha)$. For $\beta \in T_A\mathcal{A}_P = \Omega^1(M, \mathfrak{g}_P)$ by differentiating under the integral we have

$$\begin{aligned} T_A\mu^\alpha(\beta) &= \int_{\Sigma_{g,n}} (\alpha \wedge T_A F(\beta))_\kappa - \int_{\partial\Sigma_{g,n}} (\alpha \wedge T_A\varphi_B(\beta))_\kappa \\ &= \int_{\Sigma_{g,n}} (\alpha \wedge d_A\beta)_\kappa - \int_{\partial\Sigma_{g,n}} (\alpha \wedge \beta)_\kappa \end{aligned}$$

where $T_A\varphi_B(\beta) = \beta$ using the canonical identifications. So by lemma 2.1.1

$$T_A\mu^\alpha(\beta) = \int_{\Sigma_{g,n}} d(\alpha \wedge \beta)_\kappa + \int_{\Sigma_{g,n}} (d_A\alpha \wedge \beta)_\kappa - \int_{\partial\Sigma_{g,n}} (\alpha \wedge \beta)_\kappa$$

From Stokes theorem we see that

$$\int_{\Sigma_{g,n}} d(\alpha \wedge \beta)_\kappa = \int_{\partial\Sigma_{g,n}} (\alpha \wedge \beta)_\kappa$$

So finally noting that $d_A\alpha$ is the fundamental vector field associated to $\alpha \in \Omega^0(M, \mathfrak{g}_P)$ we get

$$T_A\mu^\alpha(\beta) = \int_{\Sigma_{g,n}} (d_A\alpha \wedge \beta)_\kappa = \omega_A(d_A\alpha, \beta)$$

□

Remark: If $n = 0$ then the moment map doesn't depend on the choice of φ_B and it is also equivariant with respect to the moment map.

Remark: If we give $\Sigma_{g,n}$ a complex structure then we can consider holomorphic forms $\Omega^{1,0}(\Sigma_{g,n}, \mathfrak{g}_P)$ and antiholomorphic forms $\Omega^{0,1}(\Sigma_{g,n}, \mathfrak{g}_P)$. These correspond the eigenvalues of the Hodge star operator with eigenvalues i and $-i$ respectively noting that on a surface we have $*^2 = -1$. In this sense $*$ gives $\mathcal{A}_P \cong \Omega^1(\Sigma_{g,n}, \mathfrak{g}_P)$ an almost complex structure. The Atiyah-Bott metric and symplectic form form a Kähler structure.

2.1.2 Symplectic Form on $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$

The theory of symplectic reduction can be applied to the space of connections on surface with the Hamiltonian action of $\mathcal{G}_P$. This follows a similar procedure to that outlined in section F.3 and theorem F.3.1. However dealing with the infinite dimensional smooth manifold $\mathcal{A}_P$ and a the curvature function $F : \mathcal{A}_P \to \Omega^2(M, \mathfrak{g}_P)$ which has it's image in an infinite dimensional smooth manifold leads to some complications. We are then interested in taking the quotient of $F^{-1}(0)$ under the action of $\mathcal{G}_P$ an infinite dimensional Lie group. We won't develop the analogues of the theorems from the finite dimensional theory of manifolds; however the standard theorems like the inverse function theorem, regular value theorem and Sard's theorem hold in most cases of interest. In the cases we are interested in these theorems will hold and those interested should consult section 14 of [AB83] for some discussion of the spaces in question.

The main thing that we need to check in relation to the symplectic reduction of theorem F.3.1 is that $F^{-1}(0)$ is a submanifold of $\mathcal{A}_P$. Then determining where the action of $\mathcal{G}_P$ on $F^{-1}(0)$ is free will tell us what subspace of $\mathcal{R}_{G,g,n}$ is smooth and has a symplectic form.

Lemma 2.1.5. *Let Σ be a 2-dimensional manifold, G a semi-simple Lie group and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. The regular points of the curvature map $F : \mathcal{A}_P \to \Omega^2(\Sigma, \mathfrak{g}_P)$ on $F^{-1}(0)$ are in bijection with irreducible representations under the correspondence of theorem 1.1.8.*

Proof. To show that A is a regular point we must show that $T_A F$ is surjective. Recall $T_A F = d_{A,1}$ from lemma D.3.3. This means that we want to show that $d_{A,1}$ is surjective. This amounts to showing that $im(d_{A,1}) = \Omega^2(\Sigma, \mathfrak{g}_P) = ker(d_{A,2})$. This therefore amounts to showing that $H^2(\Sigma, d_A) = 0$.

As Σ is compact by Poincare duality we want to show that $H^0(\Sigma, d_A) = 0$ if and only if A gives rise to an irreducible representation of $\pi_1(\Sigma)$. Notice that $ker(d_A) = H^0(\Sigma, d_A)$ and so $H^0(\Sigma, d_A) = 0$ if and only if $ker(d_A) = 0$.

Recall that $d_A : T_e\mathcal{G}_P = \Omega^0(\Sigma, \mathfrak{g}_P) \to \Omega^1(\Sigma, \mathfrak{g}_P) = T_A\mathcal{A}_P$ gives the infinitesimal action of the gauge group. Notice that $ker(d_A) = 0$ if and only if the infinitesimal gauge group action has zero kernel. The infinitesimal gauge group action has zero kernel if and only if the gauge group action has discrete stabiliser at A.

By lemma 1.1.9 the stabiliser of $\mathcal{G}_P$ at A is identified with the stabiliser of ρ_A with respect to G acting by conjugation. So $ker(d_A) = 0$ if and only if ρ_A has finite stabiliser with respect to the action via conjugation. As G is semi-simple the only subgroup with finite stabiliser under the action of conjugation is the whole group. Therefore the image of ρ_A is contained in no proper subgroup if and only if $ker(d_A) = 0$ which is where $T_A F$ is surjective. $\square$

Remark: Morally, this result says that a connection with holonomy representation class reducible to a connection with holonomy representation class to some subgroup isn't a connection we should consider in the smooth part of the moduli space.

Corollary 2.1.6. *(Regular Points of the Curvature Map)*
Let Σ be a 2-dimensional manifold and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. Consider the curvature and boundary map $(F, |_{\partial\Sigma}) : \mathcal{A}_P \to \Omega^2(\Sigma, \mathfrak{g}_P) \times \mathcal{R}_{\partial\Sigma,G}$ such that $(F, |_{\partial\Sigma})(A) = (F_A, [A|_{\partial\Sigma}])$ for $[B] \in \mathcal{R}_{\partial\Sigma,G}$. Then $A \in (F, |_{\partial\Sigma})^{-1}(0, [B])$ a regular point of $(F, |_{\partial\Sigma})$ if A has irreducible holonomy representation.

Corollary 2.1.7. *Let Σ be a 2-dimensional manifold and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. The action of $\mathcal{G}_P$ on the representations with irreducible holonomy representation class is free.*

> **Remark:** As mentioned in [Aud04] the singularities of the moduli space are where the curvature map F is not regular and the gauge group action is not free which makes them more severe. Goldman discusses these singularities for the finite dimensional description of the moduli space in [Gol84] and [Gol86].

Lemma 2.1.8. *(Tangent Space)*
Let $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. Then for $[A] \in \mathcal{R}_{\Sigma,G}$ we have $T_{[A]}\mathcal{R}_{\Sigma,G} \cong H^1(\Sigma, d_A)$.

Proof. We have $T_A(F^{-1}(0)) = ker(T_AF) = ker(d_{A,1})$. We also know that for $L_A : \mathcal{G}_P \to \mathcal{A}_P$ we have $im(T_AL_A) = im(d_{A,0})$. We then have

$$T_{[A]}\mathcal{R}_P = ker(d_{A,1})/im(d_{A,0}) = H^1(\Sigma, d_A)$$

□

Lemma 2.1.9. *(Poisson Structure on $\mathcal{R}_P$)*
Let Σ be a 2-dimensional manifold and $(\pi : P \to \Sigma, G, \cdot)$ be a principle bundle. Consider the adjoint bundle $ad(P) = \mathfrak{g}_P$ and its natural Riemannian metric $\eta_{P,\kappa}$ associated to $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ a positive definite inner-product invariant under the adjoint action of G on $\mathfrak{g}$.

$\mathcal{R}_P$ inherits a Poisson structure form the form $\omega^{AB,P,\kappa}$. The leaves are given by fixing the gauge equivalence class on the boundary.

Proof. We know that the action of the gauge group is Hamiltonian with moment map given by the curvature and restriction of the connection to the boundary by lemma 2.1.4. The flat connections with irreducible holonomy representation class give rise a submanifold. Noticing that where $\mathcal{G}_P$ acts freely on $(F, |_{\partial\Sigma})^{-1}(0, [B])$ is a submanifold by lemma 2.1.5 and corollary 2.1.7. We then apply the proof of theorem F.3.1 (the symplectic quotient) which only fails at showing that $(F, |_{\partial\Sigma})^{-1}(0, [B])$ is a submanifold.

This shows that fixing the flat connection on the boundary and taking the quotient gives rise to a symplectic manifold. Considering all gauge equivalence classes restricted to the boundary we get the moduli space of flat connection $\mathcal{R}_P$ and it inherits a Poisson structure as the quotient of the Poisson manifold $\mathcal{A}_{P,flat}$ by a symplectic action. The symplectic leaves are given by fixing the flat connection on the boundary. $\square$

Remark: We can use theorem 1.1.8 and instead consider the character variety. Using this we can define the symplectic form using group cohomology which has the following relation to the symplectically reduced case $H^1(\pi_1(\Sigma_{g,n}), Ad(G)) \cong H^1(\Sigma_{g,n}, d_A)$. It has the same form as the Atiyah-Bott symplectic form before it has been reduced. See [Gol84] for details. Understanding this correspondence links the natural topologies in both viewpoints. This is nice as we have a finite dimensional description for this infinite dimensional construction.

Definition: ($\mathcal{R}_{G,g,n}(C_1, ..., C_n)$)
Let $C_1, ..., C_n \in \mathcal{R}_{S^1,G} \cong G/Ad(G)$ and let $C_1 \sqcup ... \sqcup C_n = [B] \in \mathcal{R}_{\bigsqcup_{i=1}^n S^1,G}$. Let $\mathcal{P}_{g,n}$ be the isomorphism classes of principle G-bundles on $\Sigma_{g,n}$. Then let

$$\mathcal{R}_{G,g,n}(C_1, ..., C_n) = \bigsqcup_{P \in \mathcal{P}_{g,n}} (F_P, |_{\partial\Sigma})^{-1}(0, [B])/\mathcal{G}_P$$

with the symplectically reduced form $\omega_{G,g,n}(C_1, ..., C_n)$ on the dense set of smooth points.

Remark: For simple connected groups such as $SU(2)$ we have $\mathcal{P} = \{\Sigma_{g,n} \times G\}$. That is every principle G-bundle is trivial. So $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$ has only one connected component.

In the next chapter 3 we will calculate the symplectic volume of $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$ with respect to the form $\omega_{G,g,n}(C_1, ..., C_n)$.

2.2 Torus Action on $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$

In the remarkable work of Goldman [Gol86] there are canonical functions defined on the moduli space and their Hamiltonian flows calculated. Building on this, Jeffrey and Weitsman in [JW92] [JW94] have used these calculations to note that in the case of $G = SU(2)$ the flows of these functions define a half dimensional torus action. Using this and the theory of torus actions on symplectic manifolds they have calculated volumes of the moduli space.

2.2.1 Goldman's Functions

Definition: (Goldman's Functions)
Let M be smooth manifold and G a Lie group. Consider the moduli space of flat connections $\mathcal{R}_{M,G}$. Let $f : G \to \mathbb{R}$ be a class function. That is for $g, h \in G$ we have $f(ghg^{-1}) = f(h)$. Let $x \in M$ and $[\gamma] \in \pi_1(M, x)$. Define the following function called a **Goldman function**

$$f_\gamma : \mathcal{R}_{M,G} \to \mathbb{R} \quad s.t \quad f_\gamma([A]) = f(Hol_{x,\gamma}(A))$$

This is well defined as f is invariant under conjugation.

Remark: This gives us many functions on $\mathcal{R}_{M,G}$ as we have the choice of $f : G \to \mathbb{R}$ and $[\gamma] \in \pi_1(M, x)$.

Remark: These functions are pre-quantised versions of the so-called Wilson lines of Chern Simons theory.

Example: Let $G = SU(2)$ and $f = \frac{1}{\pi}\cos^{-1}(\frac{1}{2}Tr) : SU(2) \to [0, 1]$. Notice that for $\theta \in [0, 1]$ we have

$$f\left(\begin{bmatrix} e^{i\pi\theta} & 0 \\ 0 & e^{-i\pi\theta} \end{bmatrix}\right) = \theta$$

Consider $M = \Sigma_{0,3}$. Then we have

$$\mathcal{R}_{SU(2),0,3} \cong \{(X, Y, Z) \in SU(2) \times SU(2) \times SU(2) : XYZ = 1\}/SU(2)$$

Let $[\gamma_1], [\gamma_2], [\gamma_3] \in \pi_1(\Sigma_{0,3})$ be the generators with $[\gamma_1] \mapsto X$, $[\gamma_2] \mapsto Y$ and $[\gamma_3] \mapsto Z$ under the holonomy representation class for each element of $\mathcal{R}_{SU(2),0,3}$.

Then we have $f_{[\gamma_1]}([A]) = f(X)$, $f_{[\gamma_2]}([A]) = f(Y)$ and $f_{[\gamma_3]}([A]) = f(Z)$. It can therefore be seen that in this case $(f_{[\gamma_1]}, f_{[\gamma_2]}, f_{[\gamma_3]}) : \mathcal{R}_{SU(2),0,3} \to \mathbb{R}^3$ gives a homeomorphism onto it's image as described in 1.2.9.

Remark: These functions are just evaluation maps on $Hom(\pi_1(M), G)/G$ and are well defined as f is a class function.

Building on this example we have the following for surfaces.

Lemma 2.2.1. *(Boundary Values and Symplectic Leaves)*
Let Σ be smooth 2-manifold and G a connected simple Lie group. Let T be the maximal torus of G. Then there are $dim(T)$ invariant functions $\{f_1, ..., f_{dim(T)}\}$ that determine the conjugacy class of any element in G.

Consider the moduli space of flat connections $\mathcal{R}_{\Sigma,G}$. Then for each circle $\gamma \in \partial\Sigma$ the set $\{(f_1)_\gamma, ..., (f_{dim(T)})_\gamma\}$ gives coordinates for the connections restricted to γ.

Proof. Recall that in theorem 1.2.1 we proved that $\mathcal{R}_{\gamma,G} \cong G/Ad(G)$ is given by the conjugacy classes in G. Then $\{(f_1)_\gamma, ..., (f_{dim(T)})_\gamma\}$ determine the conjugacy class. □

2.2.2 Twist Flows

We will give explicit formula for the Poisson brackets and Hamiltonian flows of Goldman's functions defined in 2.2.1. For some more on these calculations consult [Gol86]. We will consider $\mathcal{R}_{G,g,n} = Hom(\pi_1(\Sigma_{g,n}), G)$ for much of this section. Notice that using theorem 1.1.8 we can interchange between our viewpoints.

Definition: (Variation Function)
Let G be a Lie group with κ a non-degenerate bilinear form on $Lie(G) = \mathfrak{g}$. Let $f : G \to \mathbb{R}$ be a class function on G. The **variation function** associated to f is defined to be $F : G \to \mathfrak{g}$ that satisfies the following property

$$\kappa(F(g), X) = \frac{d}{dt}\Big|_{t=0} f(g \exp(tX))$$

The non-degeneracy of κ means that F is defined and it will in fact be smooth.

Remark: This function has various properties such as being equivariant these can be checked by hand or one can consult section 1 of [Gol86].

Theorem 2.2.2. *(Theorem 3.5 in [Gol86])*
Let G be a Lie group with κ a non-degenerate bilinear form on $Lie(G) = \mathfrak{g}$. Let $f, f' : G \to \mathbb{R}$ be a class function on G. Let $F, F' : G \to \mathfrak{g}$ be the variation functions of f and f' respectively.

Consider $[\alpha], [\beta] \in \pi_1(\Sigma_{g,n})$ and with representatives $\alpha : [0,1] \to \Sigma_{g,n}$ and $\beta : [0,1] \to \Sigma_{g,n}$ with $\alpha[0,1]$ and $\beta[0,1]$ intersecting finitely many times and transversely. Let $\epsilon(x, \alpha, \beta) = \pm 1$ be the orientation of the intersection of $\alpha[0,1]$ and $\beta[0,1]$ at x. Let $[\alpha_x] \in \pi_1(\Sigma_{g,n}, x)$ and $[\beta_x] \in \pi_1(\Sigma_{g,n}, x)$, Then

$$\{f_\alpha, f_\beta\} : Hom(\pi_1(\Sigma_{g,n}), G)/G \to \mathbb{R} \quad s.t$$

$$\{f_\alpha, f'_\beta\}([\rho]) = \sum_{x \in \alpha[0,1] \cap \beta[0,1]} \epsilon(p, \alpha, \beta) \kappa(F(\rho[\alpha_p]), F'(\rho[\beta_p]))$$

Corollary 2.2.3. *(Corollary 3.6 in [Gol86])*
If $\alpha[0,1] \cap \beta[0,1] = \varnothing$ then

$$\{f_\alpha, f'_\beta\} = 0$$

Proof. The proof uses the identification of $T_{[A]}\mathcal{R}_{G,g,n} = H^1(\Sigma_{g,n}, \mathfrak{g}_A)$ and Poincaré duality. See section 3 of [Gol86] for details. □

Remark: We can find $3g - 3 + n$ disjoint simple curves not including the boundary circles on $\Sigma_{g,n}$. These come from a trinion decomposition. Noting lemma 2.2.1 we therefore have $(3g - 3 + n)Rank(G)$ commuting functions.

Theorem 2.2.4. *(Twist Flows Covering the Hamiltonian Flow) [Gol86]*
Let G be a Lie group with κ a non-degenerate bilinear form on $Lie(G) = \mathfrak{g}$. Let $f : G \to \mathbb{R}$ be a class function on G. Let $F : G \to \mathfrak{g}$ be the variation function of f.

Let $[\gamma] \in \pi_1(\Sigma_{g,n})$ then

- *If $\Sigma_{g,n} - \gamma[0,1] = \Sigma_1 \sqcup \Sigma_2$ is disconnected the flow generated by f_γ is covered by $\Xi_t \rho : \mathbb{R} \to Hom(\pi_1(\Sigma_{g,n}), G)$ with*

$$\Xi_t \rho(\alpha) = \begin{cases} \rho(\alpha) & \text{if } \alpha \in \pi_1(\Sigma_1) \\ \exp(tF(\rho(\gamma)))\rho(\alpha)\exp(-tF(\rho(\gamma))) & \text{if } \alpha \in \pi_1(\Sigma_2) \end{cases}$$

- *If $\Sigma_{g,n} - \gamma[0,1]$ is connected then there is a cycle corresponding to γ we'll denote β that intersects γ positively transversally once. We can then see that β and $\pi_1(\Sigma_{g,n} - \gamma[0,1])$ generated $\pi_1(\Sigma_{g,n})$. The flow generated by f_γ is covered by $\Xi_t\rho : \mathbb{R} \to Hom(\pi_1(\Sigma_{g,n}), G)$ with*

$$\Xi_t\rho(\alpha) = \begin{cases} \rho(\alpha) & \text{if } \alpha \in \pi_1(\Sigma_{g,n} - \gamma[0,1]) \\ \rho(\alpha)\exp(tF(\rho(\gamma))) & \text{if } \alpha = \beta \end{cases}$$

Proof. The proof again uses Poincaré duality. See section 4 of [Gol86] for details. □

The flows of the functions associated to $3g-3+n$ disjoint curves defines the action of $\mathbb{R}^{Rank(G)(3g-3+n)}$ on the moduli space. We are interested in whether this action is periodic which will imply the flows will be determined by a $U(1)^{Rank(G)(3g-3+n)}$ action.

Lemma 2.2.5. *(SU(2)-Periodic Flows)*
Let $\kappa : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathbb{R}$ be the killing form defined by $\kappa(X,Y) = Tr(XY)$. Let $f = Tr : SU(2) \to \mathbb{R}$. Then for $[\gamma] \in \pi_1(\Sigma_{g,n})$ the Hamiltonian flow of the function $f_\gamma : \mathcal{R}_{SU(2),g,n} \to \mathbb{R}$ denoted $\Xi_t\rho$ is periodic.

Proof. By the explicit description of the Hamiltonian flows given by Goldman in theorem 2.2.4. We will need only show that $\exp(tF(\rho(\gamma)))$ is periodic. Goldman in section 1 of [Gol86] calculates $F(g) = \frac{1}{2}\left(g - g^{-1}\right)$ where we consider $g \in SU(2) \subseteq M_2(\mathbb{C})$.

Now for $\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} \in SU(2)$ we have

$$F\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} - \frac{1}{2}\begin{bmatrix} \overline{a} & -b \\ \overline{b} & a \end{bmatrix} = i\begin{bmatrix} Im(a) & 0 \\ 0 & -Im(a) \end{bmatrix}$$

So we see that

$$\exp\left(tF\left(\begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}\right)\right) = \begin{bmatrix} \exp(itIm(a)) & 0 \\ 0 & \exp(itIm(a)) \end{bmatrix}$$

This is periodic with period $\frac{2\pi}{Im(a)}$ or if $Im(a) = 0$ is fixed.

Therefore the flow of the function f_γ is periodic. □

Corollary 2.2.6. *($\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$ is a Toric Variety)*
Let for $C_1, ..., C_n \in \mathcal{R}_{S^1,SU(2)} \cong SU(2)/Ad(SU(2))$ let $\mathcal{R}_{SU(2),g,n}(C_1, ..., C_n)$ be the moduli space of flat connections with boundary specified by $C_1, ..., C_n$ for each boundary circle.

Let $Trin$ be a pair of pants decomposition with circles $\Gamma = \{\gamma_1, ..., \gamma_{3g-3+n}\}$. Note that the elements of Γ are disjoint simple closed curves on $\Sigma_{g,n}$. Consider the $Rank(SU(2)) = 1$ invariant function given by $f = \frac{1}{\pi}\cos^{-1}\left(\frac{1}{2}Tr\right) : SU(2) \to [0,1]$.

Then the flow of the functions $f_{\gamma_1}, ..., f_{\gamma_{3g-3+n}}$ defines the action of $U(1)^{3g-3+n}$ on the moduli space $\mathcal{R}_{SU(2),g,n}(C_1, ..., C_n)$. Moreover $\mathcal{R}_{SU(2),g,n}(C_1, ..., C_n)$ is a toric variety.

Proof. We have the action of $U(1)^{3g-3+n}$ on $\mathcal{R}_{SU(2),g,n}(C_1,...,C_n)$ is Hamiltonian with moment map given by

$$\mu = (f_{\gamma_1},...,f_{\gamma_{3g-3+n}}) : \mathcal{R}_{SU(2),g,n}(C_1,...,C_n) \to \mathbb{R}^{3g-3+n}$$

and the dimension of $\mathcal{R}_{SU(2),g,n}(C_1,...,C_n)$ is

$$\dim\left(Hom(\pi_1(\Sigma_{g,n}, SU(2))/SU(2)\right) - nRank(SU(2))$$

$$= (2g+n-1)\dim(SU(2)) - \dim(SU(2)) - n = 6g-6+2n = 2(3g-3+n)$$

So the torus action is half dimensional. □

Remark: Corollary 2.2.6 is not valid for general G and is a special property of $SU(2)$. For example for general g, n we have

$$\dim\left(Hom(\pi_1(\Sigma_{g,n}, SU(3))/SU(3)\right) - nRank(SU(3))$$

$$= (2g+n-2)\dim(SU(3)) - 2n = 16g-16+6n$$

$$\neq 12g-12+4n = 2Rank(SU(3))(3g-3+n)$$

Chapter 3

Volumes of the Moduli Space of Flat Connections

In [Wit91] Witten uses three different methods to calculate the volume of the moduli space of flat connections over a surface with a semi-simple compact connected Lie group G. This is done using partition functions in QFT as well as using the Verlinde formula in CFT for $SU(2)$. Witten rigorously calculates the volume through the theory of Reidemeister torsion which is found to be explicitly related to the symplectic volume.

In this section we will discuss a fourth way of calculating the volumes for $SU(2)$ using symplectic geometry via the techniques described in [JW94]. We will then consider Witten's general formula and some of it's properties. Finally we will make some remarks on how these volumes are related to the work of Mirzakhani on the Weil-Petterson volumes of the moduli space of curves.

3.1 The Case of $SU(2)$

To calculate the volume of the moduli space we will use the fact the moduli space of flat $SU(2)$ connections on a surface with boundary holonomy specified is a toric variety. This was shown in corollary 2.2.6. The Duistermaat-Heckman theorem then reduces the calculation to calculating a particular Euclidean volume. Calculating these Euclidean volumes is done using lattice point counts and recursions.

3.1.1 Convex Polyhedra and Volumes

We will specify some conventions in the following definitions.

Definition: $(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n))$
Let $t_i \in [0, 1]$ and define $\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$ to be $\mathcal{R}_{SU(2),g,n}(C_1, ..., C_n)$ with $C_i \in \mathcal{R}_{S^1,SU(2)} \cong SU(2)/Ad(SU(2))$ such that

$$C_i = \begin{bmatrix} e^{i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{bmatrix}$$

Recall that from corollary 2.2.6 $\mathcal{R}_{SU(2),g,n}(t_1,...,t_n)$ is a toric variety with moment map given Goldman's functions associated to a trinion decomposition.

Definition: $(V_{g,n,SU(2)})$
Let $V_{g,n,SU(2)}(t_1,...,t_n) = Vol(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n))$.

Our goal is to calculate and understand the relations satisfied by the functions $V_{g,n,SU(2)}(t_1,...,t_n)$.

Lemma 3.1.1. *($V_{g,n,SU(2)}(t_1,...,t_n)$)*
Let $f = \frac{1}{\pi}\cos^{-1}\left(\frac{1}{2}Tr\right) : SU(2) \to [0,1]$ and $\gamma_1,...,\gamma_{3g-3+n}$ be non-intersecting simple closed loops. Recall from corollary 2.2.6 the Hamiltonian $U(1)^{3g-3+n}$-action on $\mathcal{R}_{SU(2),g,n}(t_1,...,t_n)$ with moment map

$$\mu = (f_{\gamma_1},...,f_{\gamma_{3g-3+n}}) : \mathcal{R}_{SU(2),g,n}(t_1,...,t_n) \to \mathbb{R}^{3g-3+n}$$

By the Duistermaat-Heckman theorem and its corollary F.4.3 we see that

$$V_{g,n,SU(2)}(t_1,...,t_n) = 2^{-2g+3-n} Vol_{Euc}(\mu(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n)))$$

Remark: To see why we need the factor of 2 see proposition 3.10 in [JW94]. This makes sure we calculate the right Euclidean volume described in corollary F.4.3.

This means that calculating the volume reduces to calculating the volume of the image of the moment map. Notice that the $3g-3+n$ curves decompose the surface $\Sigma_{g,n}$ into $2g-2+n$ pairs of pants. Therefore understanding the pair of pants will be our starting point.

Recalling the homomorphism described in section 1.2.9 we have the following result for the volume of $V_{0,3}(t_1,t_2,t_3)$.

Lemma 3.1.2. *(Pair of Pants)*

$$V_{0,3,SU(2)}(t_1,t_2,t_3) = \begin{cases} 1 & \textit{if } |t_1-t_2| \leqslant t_3 \leqslant \min(t_1+t_2, 2-t_1-t_2) \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Notice that $\mathcal{R}_{SU(2),g,n}(t_1,t_2,t_3)$ is $6\times 0 - 6 + 2n = 0$-dimensional. Therefore we have

$$Vol_{Euc}(\mu(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n))) = \begin{cases} 1 & \text{if } \mathcal{R}_{SU(2),g,n}(t_1,...,t_n) \neq \varnothing \\ 0 & \text{it } \mathcal{R}_{SU(2),g,n}(t_1,...,t_n) = \varnothing \end{cases}$$

From the homomorphism described in lemma 1.2.9 we see that

$$\mathcal{R}_{SU(2),g,n}(t_1,...,t_n) \neq \varnothing \quad \text{if and only if} \quad |t_1-t_2| \leqslant t_3 \leqslant \min(t_1+t_2, 2-t_1-t_2)$$

∎

Theorem 3.1.3. *(Volume of the Moduli Space of Flat $SU(2)$ connections)*
Let $\Gamma = \{\gamma_1, ..., \gamma_{3g-3+2n}\}$ be a collection of non-intersecting simple closed curves that induce a pair of pants decomposition denoted by $Trin$ with boundary circle given by $\gamma_{3g-3+n+1}, ..., \gamma_{3g-3+n+n}$. We then have the following

$$V_{g,n}(t_1, ..., t_n) = 2^{-2g+3-n} Vol_{Euc}\Big(\Big\{(x_1, ..., x_{3g-3+n}) \in \mathbb{R}^{3g-3+n} : \textit{for } T \in Trin$$

$$\textit{and } \gamma_i, \gamma_j \gamma_k \in T \textit{ we have } |x_i - x_j| \leqslant x_k \leqslant \min(x_i + x_j, 2 - x_i - x_j) \textit{ where}$$

$$t_1 = x_{3g-3+n+1}, ..., t_n = x_{3g-3+n+n}\Big\}\Big)$$

Proof. By the Duistermaat-Heckman theorem we have

$$V_{g,n,SU(2)}(t_1, ..., t_n) = 2^{-2g+3-n} Vol_{Euc}(\mu(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)))$$

Now by lemma 1.2.7 we can see that the image of the moment map

$$\mu(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n))$$

is given exactly by considering the aloud holonomies around each γ with respect to the pairs of pants they bound. The holonomies of the boundary must also match our boundary conditions. This describes all the conditions on the elements of $\mu(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n))$ and these are exactly the conditions given in the set

$$\Big\{(x_1, ..., x_{3g-3+n}) \in \mathbb{R}^{3g-3+n} : \text{for } T \in Trin \text{ and } \gamma_i, \gamma_j\gamma_k \in T \text{ we have}$$

$$|x_i - x_j| \leqslant x_k \leqslant \min(x_i + x_j, 2 - x_i - x_j) \text{ where } t_1 = x_{3g-3+n+1}, ..., t_n = x_{3g-3+n+n}\Big\}$$

□

Remark: We can see the Atiyah-Guillemin-Sternberg convexity theorem coming into play here as we notice that the image of the moment map is a convex polyhedron.

3.1.2 Lattice Point Counts

In the theory of geometric quantisation, one takes a line bundle $\mathcal{L} \to \mathcal{R}_{SU(2),g,n}$ with $c_1(\mathcal{L}) = \omega$ where ω is the symplectic form on $\mathcal{R}_{SU(2),g,n}$. One then imposes a complex or symplectic structure and takes the vector space of holomorphic or covariant constant sections as the Hilbert space of our quantum system.

There is a geometric quantisation of the moduli space of flat connections on a surface. On one hand, we have the so-called complex polarisation which leads to the complex algebro-geometric approach to the subject. This is the viewpoint of [Tha95]. On the other hand, with the symplectic set up we have described by taking a so-called real polarization, we get the symplectic case which is the viewpoint in [JW92]. In this case, the covariant constant sections are in bijection with special integral points in the image of the moment map associated to Goldman's torus action described in section 2.2.4.

We will not be interested in any geometric quantisation of the moduli space; however we will be interested in these integral points in the image of the moment of Goldman's torus action described in section 2.2.4. These integral points can calculate the Euclidean volume of the image of the moment map. Calculating Euclidean volumes of reasonable subsets can be done by taking a lattice with some characteristic distance $\frac{1}{k}$ in Euclidean space. In general, the number of points in the set will be proportional to the volume in the highest degree terms of k. The volume can be calculated by taking a limit. This will be the method described in this section.

Lemma 3.1.4. *(Lattice Point Counts)*
Let $\Delta \subseteq \mathbb{R}^n$ be a compact convex polyhedron. Then we have

$$\lim_{k\to\infty} \frac{\#\left(\frac{1}{k}\mathbb{Z}^n \cap \Delta\right)}{k^n} = Vol_{Euc}(\Delta)$$

We will use this lemma to calculate the volume of $\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$.

Definition: (Lattice Points)
Consider $\mu = (f_{\gamma_1}, ..., f_{\gamma_{3g-3+n}}) : \mathcal{R}_{SU(2),g,n}(t_1, ..., t_n) \to \mathbb{R}^{3g-3+n}$. The set of **lattice points** k are given by the set

$$B_{g,n,k}(t_1, ..., t_n) = \frac{1}{k}\mathbb{Z}^{3g-3+n} \cap \mu\left(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)\right)$$

This is of interest as this is exactly the set of lattice points in lemma 3.1.4 that allows us to calculate the volume as $k \to \infty$.

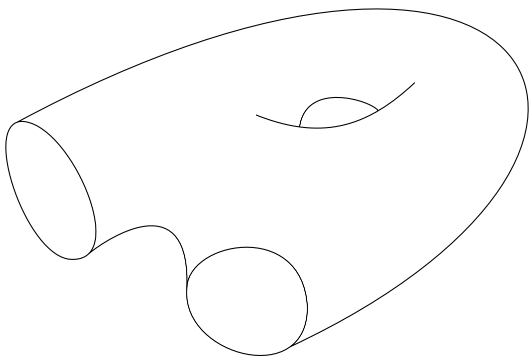
Colourings of Trivalent Graphs: Surfaces can be represented pictorially by trivalent graphs. The process for doing this is

- Take a pair of pants decomposition
- At for each trinion $T_1, ..., T_{2g-2+n}$ take a vertex $v_1, ..., v_{2g-2+n}$.
- For each circle $\gamma_1, ..., \gamma_{3g-3+n}$ if γ_i bounds two trinions T_j and T_k we take an edge e_i connecting the vertices v_j and v_k
- For a boundary circle $\gamma_{3g-3+n+1}, ..., \gamma_{3g-3+n+n}$ if γ_i bounds a trinion T_j take a half edge from v_j.

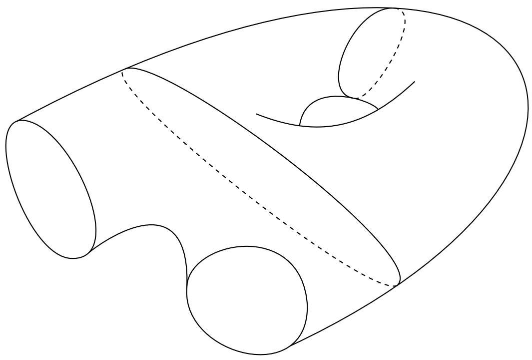
Remark: This could be stated formally as a bijection between isotopy classes of pair of pants decompositions and trivalent graphs with half edges. Therefore, this gives a well defined map from trivalent graphs to surfaces but not in the other direction. This is simply stating that pair of pants decompositions are not unique.

This process illustrated in the following examples.

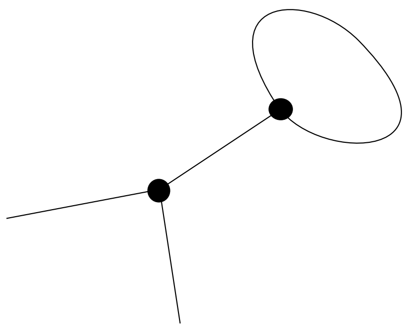
Example: ($\Sigma_{1,2}$) The surface $\Sigma_{1,2}$ is depicted as follows.



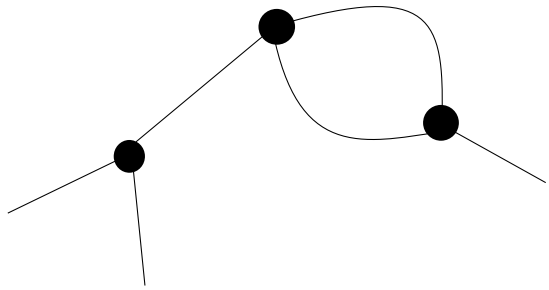
Take the following pair of pants decomposition.



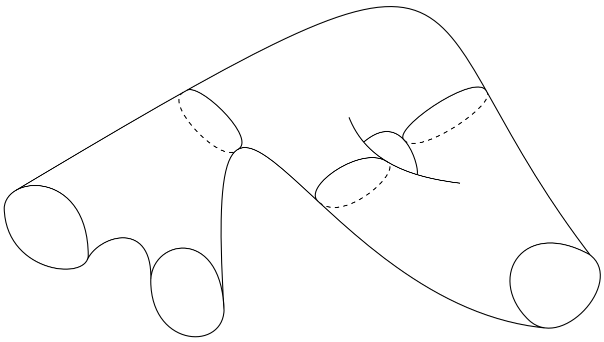
We have two pairs of pants so we get two vertices. The two pairs of pants are glued along one circle, so we get one edge between the two vertices. One pair of pants has two boundary components, so we get two half edges attached to one vertex. The other pair of pants is glued along two of its edges, so we get an edge from the pair of pants to itself.



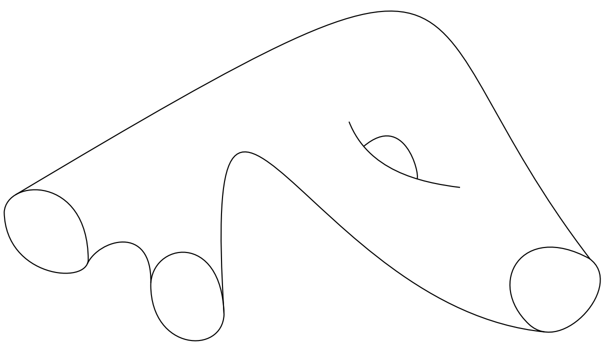
Example: ($\Sigma_{1,3}$) Consider the following trivalent graph.



We take one pair of pants for each vertex. Therefore, we take three pairs of pants. The left most vertex has two half edges; therefore, the pair of pants associated to this vertex has two boundary components. It is then connected to the middle pair of pants by one boundary. The other two circles in the middle pair of pants are then attached to the right most vertex. The right most vertex then has one half edge which becomes a boundary circle.



Forgetting the pair of pants decomposition, we have the following surface.



These pictures can be used to carry out the calculation the number of lattice points

$$\#B_{g,n,k}(t_1,...,t_n) = \#\left(\frac{1}{k}\mathbb{Z}^{3g-3+n} \cap \mu\left(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n)\right)\right)$$

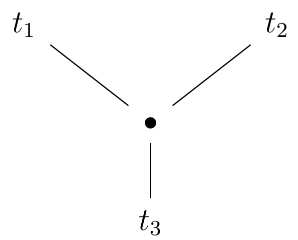
These calculations are reminiscent of calculations associated to Feynmann diagrams. The calculation proceeds as follows.

- Take a trivalent graph representing $\Sigma_{g,n}$
- Take all possible labellings $x_1,...,x_{3g-3+n}$ of the internal edges $e_1,...,e_{3g-3+n}$ with $x_i \in \frac{1}{k}\mathbb{Z} \cap [0,1]$ and label the external half edges by $t_1,...,t_n$
- We then count all the colourings $x_1,...,x_{3g-3+n}$ such that each a vertex v with edges e_i, e_j, e_k with labels $x_i, x_j, x_k \in \frac{1}{k}\mathbb{Z} \cap [0,1]$ we have

$$|x_i - x_j| \leqslant x_k \leqslant \min(x_i + x_j, 2 - x_i - x_j)$$

We can see from theorem 3.1.3 that this count precisely gives the number $\#B_{g,n,k}(t_1,...,t_n)$. We will flesh out this process in some simple examples.

Example: $(\#B_{0,3,k}(t_1,t_2,t_3) = \#\left(\mu\left(\mathcal{R}_{SU(2),0,3}(t_1,t_2,t_3)\right)\right))$
The following trivalent graph is associated to $\Sigma_{0,3}$.



This is the trivial example as we have no internal edges. Therefore the number of colourings is given as follows

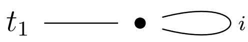
$$\#B_{0,3,k}(t_1,t_2,t_3) = V_{0,3,SU(2)}(t_1,t_2,t_3) = \begin{cases} 1 & \text{if } |t_1 - t_2| \leqslant t_3 \leqslant \min(t_1 + t_2, 2 - t_1 - t_2) \\ 0 & \text{otherwise} \end{cases}$$

Recall that this is symmetric such that for $\sigma \in S_3$ we have

$$\#B_{0,3,k}(t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)}) = \#B_{0,3,k}(t_1,t_2,t_3)$$

> **Remark:** This example is important as, to count the number of lattice points, one simply sums over all colourings with a weight given by $\#B_{0,3,k}(t_1,t_2,t_3) = V_{0,3,SU(2)}(t_1,t_2,t_3)$ at each vertex. This will give zero for the colourings we don't want and one for the colourings we want. One should compare with theorem 3.1.3 and note that this is a direct consequence.

Example: $(\#B_{1,1,k}(t_1) = \#\left(\mu\left(\mathcal{R}_{SU(2),1,1}(t_1)\right)\right))$
The following trivalent graph is associated to $\Sigma_{1,1}$.



Consider all the colourings of the edges of this graph. There is only one internal edge that can be coloured. Therefore

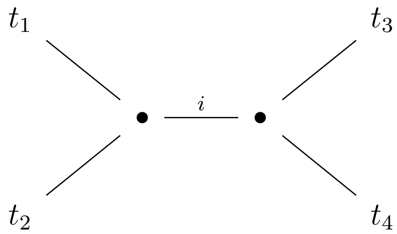
$$\#B_{1,1,k}(t_1) = \sum_{i=0}^{k} V_{0,3,SU(2)}\left(t_1, \frac{i}{k}, \frac{i}{k}\right) = \sum_{i=0}^{k} \begin{cases} 1 & \text{if } |\frac{i}{k} - \frac{i}{k}| \leqslant t_1 \leqslant \min(\frac{i}{k} + \frac{i}{k}, 2 - \frac{i}{k} - \frac{i}{k}) \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{i=0}^{k} \begin{cases} 1 & \text{if } 0 \leqslant kt_1 \leqslant \min(2i, 2k-2i) \\ 0 & \text{otherwise} \end{cases} = \sum_{\frac{kt_1}{2} \leqslant i \leqslant k - \frac{kt_1}{2}} 1 = k - \frac{kt_1}{2} - \frac{kt_1}{2} + 1 = k(1-t_1)+1$$

Extracting the top coefficient and using the appropriate factors of 2 we see that

$$V_{1,1,SU(2)}(t_1) = \frac{1}{2}(1-t_1)$$

Example: $(\#B_{0,4,k}(t_1,t_2,t_3,t_4) = \#\left(\frac{1}{k}\mathbb{Z} \cap \mu\left(\mathcal{R}_{SU(2),0,4}(t_1,t_2,t_3,t_4)\right)\right))$
The following trivalent graph is associated to $\Sigma_{0,4}$.



Consider all the colourings of the edges of this graph. There is only one internal edge we can colour. We therefore have

$$\#B_{0,4,k}(t_1,t_2,t_3,t_4) = \sum_{i=0}^{k} V_{0,3,SU(2)}\left(t_1, t_2, \frac{i}{k}\right) V_{0,3,SU(2)}\left(\frac{i}{k}, t_3, t_4\right)$$

Notice that $V_{0,3,SU(2)}\left(t_1, t_2, \frac{i}{k}\right) V_{0,3,SU(2)}\left(\frac{i}{k}, t_3, t_4\right) \neq 0$ if and only if

$$\max(|t_1-t_2|, |t_3-t_4|) \leqslant \frac{i}{k} \leqslant \min\left(\min(t_1+t_2, 2-t_1-t_2), \min(t_3+t_4, 2-t_3-t_4)\right)$$

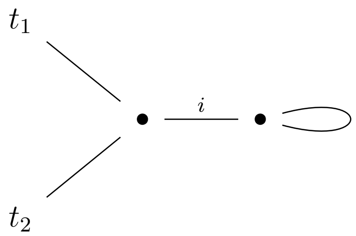
Therefore

$$\begin{aligned} &\#B_{0,4,k}(t_1,t_2,t_3,t_4) \\ &= k \max\Big[\min\left(\min(t_1+t_2, 2-t_1-t_2), \min(t_3+t_4, 2-t_3-t_4)\right) \\ &\qquad - \max(|t_1-t_2|, |t_3-t_4|), 0\Big] + 1 \end{aligned}$$

Extracting the top coefficient and using the appropriate factors of 2 we see that

$$\begin{aligned} V_{0,4,SU(2)}(t_1,t_2,t_3,t_4) = \frac{1}{2} \max\Big[&\min\left(\min(t_1+t_2, 2-t_1-t_2), \min(t_3+t_4, 2-t_3-t_4)\right) \\ &- \max(|t_1-t_2|, |t_3-t_4|), 0\Big] \end{aligned}$$

Example: $\left(\#B_{1,2,k}(t_1,t_2) = \#\left(\frac{1}{k}\mathbb{Z} \cap \mu\left(\mathcal{R}_{SU(2),1,2}(t_1,t_2)\right)\right)\right)$
The following trivalent graph is associated to $\Sigma_{1,2}$.



Consider all the colourings of the edges of this graph. The calculation of $\#B_{1,1,k}(t_1)$ will aid the calculation of $\#B_{1,2,k}(t_1,t_2)$. Take sums over the edge i and weight with now not only $\#B_{0,3,k}\left(t_1,t_2,\frac{i}{k}\right)$ but $\#B_{1,1,k}\left(\frac{i}{k}\right)$. We see that

$$\#B_{1,2,k}(t_1,t_2) = \sum_{i=0}^{k} \#B_{0,3,k}\left(t_1,t_2,\frac{i}{k}\right)\#B_{1,1,k}\left(\frac{i}{k}\right)$$

$$= \sum_{k|t_1-t_2|\leqslant i\leqslant k\min(t_1+t_2,2-t_1-t_2)} \#B_{1,1,k}\left(\frac{i}{k}\right) = \sum_{k|t_1-t_2|\leqslant i\leqslant k\min(t_1+t_2,2-t_1-t_2)} (k-i+1)$$

$$= (k+1)\left(k\min(t_1+t_2,2-t_1-t_2) - k|t_1-t_2|\right)$$

$$-\frac{1}{2}\left(k\min(t_1+t_2,2-t_1-t_2)\right)\left(k\min(t_1+t_2,2-t_1-t_2)+1\right)$$

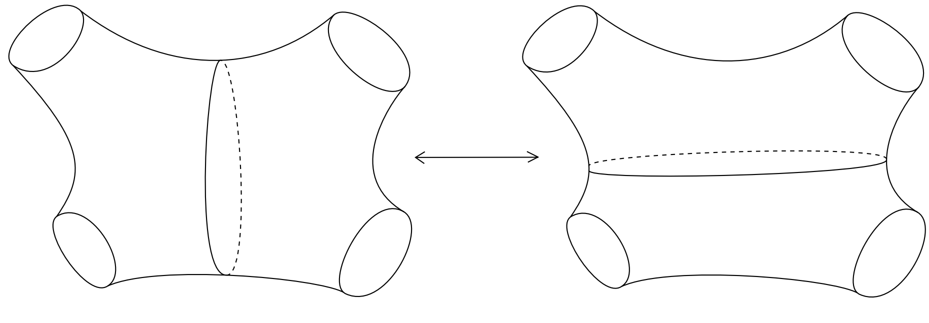
$$+\frac{1}{2}\left(k|t_1-t_2|\right)\left(k|t_1-t_2|-1\right)$$

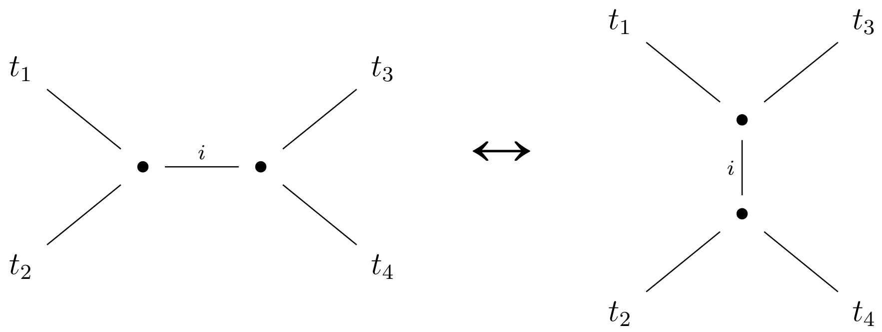
$$= k^2\left(t_1+t_2-2t_1t_2-|t_1-t_2|\right) + k\left(\frac{1}{2}\min(t_1+t_2,2-t_1-t_2) - \frac{3}{2}|t_1-t_2|+1\right)+1$$

The last equality takes a little work but can be done on paper. Extracting the top coefficient and using the appropriate factors of 2 we see that

$$V_{1,2}(t_1,t_2) = \frac{1}{4}t_1 + \frac{1}{4}t_2 - \frac{1}{2}t_1t_2 - \frac{1}{4}|t_1-t_2|$$

> **Remark:** These lattice point counts a priori depend on the choice of trinion decomposition. It can be shown at least when $t_i \in \frac{1}{k}\mathbb{Z} \cap [0,1]$ that this is independent of the trinion decomposition. Noting that all trinion decompositions are related by moves of the following form





To check that this is well defined one only needs to check that the calculation of $\#B_{0,4,k}(t_1, t_2, t_3, t_4)$ is independent of trinion decomposition. This can be done explicitly with the formula calculated for $\#B_{0,4,k}(t_1, t_2, t_3, t_4)$.

Remark: These kinds of calculations resemble calculations in 2 dimensional TQFT. In fact for a fixed k there is an underlying 2 dimensional TQFT associated to the number $\#B_{g,n,k}(t_1, ..., t_n)$. This is related to the quantisation of the moduli space. This can be seen either through the quantisation through the real polarization of Jeffrey and Weitsman in [JW92] or via the complex polarization which gives us the $SU(2)$-WZW model in CFT described in [Bea94]. In particular given a Riemann surface there is an associated vector space and $\#B_{g,n,k}(t_1, ..., t_n)$ calculates the dimension of this vector space for $t_i \in \frac{1}{k}\mathbb{Z} \cap [0, 1]$.

Calculating general formulas for these dimensions in terms of the algebraic geometry associated to the WZW CFT is quite hard. However from some gluing techniques in CFT, Verlinde in [Ver88] conjectured a solution to this problem. A proof can be found for example in [Bea94].

Definition: (Fusion Algebra for $SU(2)$)
$\mathcal{V}_{SU(2),k} = span_{\mathbb{C}}\{v_0, v_1, ..., v_k\}$. Define the following metric

$$\langle v_i, v_j \rangle = \#B_{0,2,k}(v_i, v_j) = \delta_{ij}$$

Define the following product

$$v_i \cdot v_j = \sum_{l=0}^{k} \#B_{0,3,k}\left(\frac{i}{k}, \frac{j}{k}, \frac{l}{k}\right) v_l$$

$\mathcal{V}_{SU(2),k}$ is a commutative Frobenius algebra.

Proof. The main thing to check is that the algebra is associative. One must check that $\#B_{0,4}(t_1, t_2, t_3, t_4)$ is independent of the choice of trinion decomposition. This can be done explicitly with the formula we proved for $\#B_{0,4}(t_1, t_2, t_3, t_4)$. □

Remark: The Frobenius algebra computes all the $\#B_{g,n,k}(t_1, ..., t_n)$ for $t_i \in \frac{1}{k}\mathbb{Z} \cap [0, 1]$. This in fact defines a TQFT.

Theorem 3.1.5. *(Verlinde's Formula for $SU(2)$)*
For $t_i \in \frac{1}{k}\mathbb{Z} \cap [0,1]$ we have

$$\#B_{g,n,k}(t_1,...,t_n) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{i=0}^{k} \frac{\sin\left(\frac{\pi(i+1)(kt_1+1)}{k+2}\right)...\sin\left(\frac{\pi(i+1)(kt_n+1)}{k+2}\right)}{\sin\left(\frac{\pi(i+1)}{k+2}\right)}$$

Proof. Verlinde uses gluing arguments in CFT to show that the matrix given for $i,j \in \{0,...,k\}$ as

$$S_{ij} = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi(i+1)(j+1)}{k+2}\right)$$

gives the change of coordinates from $v_0,...,v_k$ to a basis of idempotents. Then using some basic calculations in Frobenius algebras we find that

$$\#B_{g,n,k}(t_1,...,t_n) = \sum_{i=0}^{k} \frac{S_{i,kt_1}...S_{i,kt_n}}{S_{0i}^{2g-2+n}}$$

Consult [Bea94] for a discussion on some of the representation theory involved. See section 4 on curve operators in [AU06] for a rigorous discussion of the gluing techniques which gives rise to the matrix S. This is then neatly summarised in [ABO15]. □

> **Remark:** Notice that $\#B_{g,n,k}(t_1,...,t_n)$ is an integer which makes Verlinde's formula even more remarkable.

3.1.3 Recursions for Volumes

The torus action of Goldman and the Duistermaat-Heckman theorem have determined an expression for the volume of the moduli space of flat connections. There is an explicit formula for the volume $V_{0,3,SU(2)}(t_1,t_2,t_3)$ which is piecewise polynomial in t_1,t_2,t_3. As demonstrated in the previous section 3.1.2, we can calculate some of the volumes associated to the surfaces with lower Euler characteristics using our knowledge of the surfaces with larger Euler characteristic. These recursions are described formally in this section.

To calculate Euclidean volumes of a subset of $\mathbb{R}^n$, one integrates the character function associated to the subset.

Lemma 3.1.6. *(Character Functions for the Delzant Polyhedron of the Pair of Pants)*

$$\chi_{\mu(\mathcal{R}_{SU(2),0,3}(t_1,t_2,t_3))} = V_{0,3}(t_1,t_2,t_3)$$

Lemma 3.1.7. *(Character Functions for the Delzant Polyhedron for Goldman's Torus Action)*
Let $Trin$ be a pair of pants decomposition with circles $\Gamma = \{\gamma_1,...,\gamma_{3g-3+2n}\}$ and boundary circles given by $\{\gamma_{3g-3+n+1},...,\gamma_{3g-3+n+n}\}$. Let the circles bounding trinion T be given by $T(1),T(2),T(3) \in \Gamma$. Let $x_{3g-3+n+i} = t_i$. Then

$$\chi_{\mu(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n))} = \prod_{T \in Trin} V_{0,3}(x_{T(1)},x_{T(2)},x_{T(3)})$$

Proof. This follows from theorem 3.1.3 which describes the set $\mu(\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n))$ as follows.

$$\Big\{(x_1, ..., x_{3g-3+n}) \in \mathbb{R}^{3g-3+n} : \text{for } T \in Trin \text{ a } \gamma_i, \gamma_j\gamma_k \in T \text{ we have}$$

$$|x_i - x_j| \leqslant x_k \leqslant \min(x_i + x_j, 2 - x_i - x_j) \text{ where } t_1 = x_{3g-3+n+1}, ..., t_n = x_{3g-3+n+n}\Big\}$$

□

Now using this expression for the character function of the image of the Goldman moment map, we can immediately see the following kinds of recursions for the volume of the moduli space.

Corollary 3.1.8. *(Recursions for the Volume)*

$$2V_{g,n,SU(2)}(t_1, ..., t_n) = \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, x, y) V_{g-1,n+1,G}(x, y, t_2, ..., t_n) dxdy$$

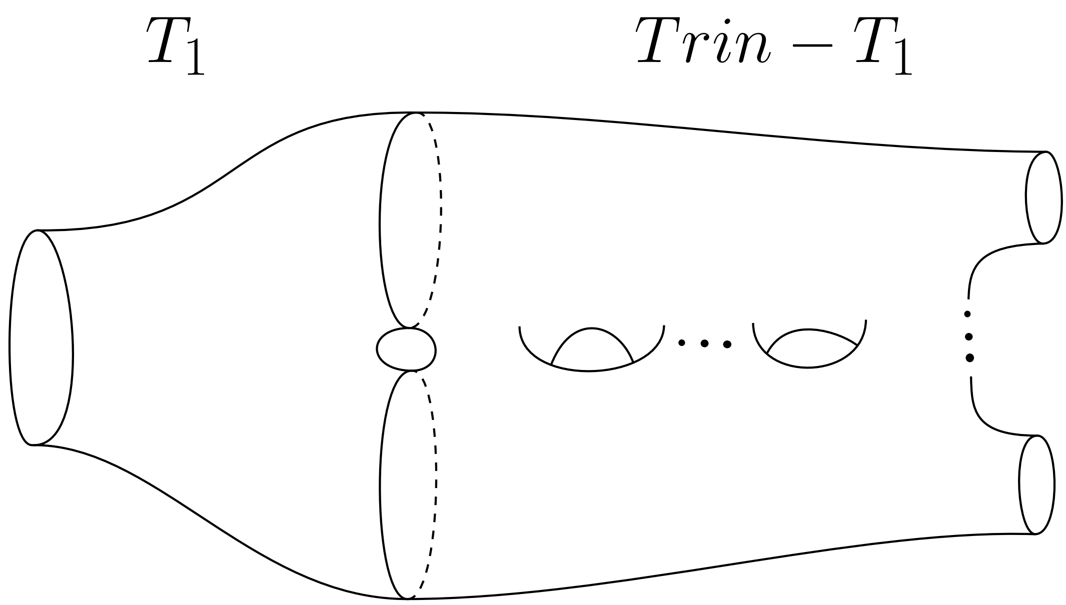
For $g_1 + g_2 = g$ and $1 + n_1 + n_2 = n$ we have

$$2V_{g,n,SU(2)}(t_1, ..., t_n)$$

$$= \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, x, y) V_{g_1,n_1+1,SU(2)}(x, t_2, ..., t_{n_1+1}) V_{g_1,n_1+1,SU(2)}(y, t_{n_1+2}, ..., t_n) dxdy$$

$$2V_{g,n,SU(2)}(t_1, ..., t_n) = \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, t_2, x) V_{g,n-1,SU(2)}(x, t_3, ..., t_n) dx$$

Proof. We'll just prove the first equality and the rest follow a similar argument. Take a pair of pants decomposition of the follow form.



Let $x_T = (x_{T(1)}, x_{T(2)}, x_{T(3)})$. Then we have

$$V_{g,n,SU(2)}(t_1, ..., t_n) = 2^{-2g+3-n} \int_{\mathbb{R}^{3g-3+n}} \chi_{\mu(\mathcal{R}_{SU(2),g,n}(t_1,...,t_n))}$$

$$= 2^{-2g+3-n} \int_0^1 ... \int_0^1 \prod_{T \in Trin} V_{0,3}(x_T) dx_1 ... dx_{3g-3+n}$$

$$= 2^{-1} \int_0^1 \int_0^1 V_{0,3}(x_{T_1}) 2^{-2(g-1)+3-(n+1)} \int_0^1 ... \int_0^1 \prod_{T \in Trin - T_1} V_{0,3}(x_T) dx_1 ... dx_{3g-3+n}$$

Notice $Trin - T_1$ is trinion decomposition of $\Sigma_{g-1,n+1}$. Therefore we see that

$$2^{-2(g-1)+3-(n+1)} \int_0^1 ... \int_0^1 \prod_{T \in Trin - T_1} V_{0,3}(x_T) dx_1 ... \widehat{dx_{T_1(2)}} ... \widehat{dx_{T_1(3)}} ... dx_{3g-3+n}$$

$$= V_{g-1,n+1,G}(x_{T(2)}, x_{T(3)}, t_2, ..., t_n)$$

Therefore we have

$$V_{g,n,SU(2)}(t_1, ..., t_n) = 2^{-1} \int_0^1 \int_0^1 V_{0,3}(x_{T_1}) V_{g-1,n+1,G}(x_{T(2)}, x_{T(3)}, t_2, ..., t_n) dx_{T_1(2)} dx_{T_1(3)}$$

□

This can be restated in terms of the lattice point counts as follows.

Lemma 3.1.9. *(Recursions for the Lattice Point Counts)*

$$\#B_{g,n,k}(t_1, ..., t_n) = \sum_{i,j=0}^{k} \#B_{0,3,k}\left(t_1, \frac{i}{k}, \frac{j}{k}\right) \#B_{g-1,n+1,k}\left(\frac{i}{k}, \frac{j}{k}, t_2, ..., t_n\right)$$

For $g_1 + g_2 = g$ and $1 + n_1 + n_2 = n$ we have

$$\#B_{g,n,k}(t_1, ..., t_n)$$

$$= \sum_{i,j=0}^{k} \#B_{0,3,k}\left(t_1, \frac{i}{k}, \frac{j}{k}\right) \#B_{g_1,n_1+1,k}\left(\frac{i}{k}, t_2, ..., t_{n_1+1}\right) \#B_{g_2,n_2+1,k}\left(\frac{j}{k}, t_{n_1+2}, ..., t_n\right)$$

$$\#B_{g,n,k}(t_1, ..., t_n) = \sum_{i=0}^{k} \#B_{0,3,k}\left(t_1, t_2, \frac{i}{k}\right) \#B_{g,n-1,k}\left(\frac{i}{k}, t_3, ..., t_n\right)$$

Corollary 3.1.10. *(Recursions for the Volume from Verlinde's Formula)*

$$2V_{g,n,SU(2)}(t_1, ..., t_n) = \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, x, y) V_{g-1,n+1,G}(x, y, t_2, ..., t_n) dx dy$$

For $g_1 + g_2 = g$ and $1 + n_1 + n_2 = n$ we have

$$2V_{g,n,SU(2)}(t_1, ..., t_n)$$

$$= \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, x, y) V_{g_1,n_1+1,SU(2)}(x, t_2, ..., t_{n_1+1}) V_{g_1,n_1+1,SU(2)}(y, t_{n_1+2}, ..., t_n) dx dy$$

$$2V_{g,n,SU(2)}(t_1, ..., t_n) = \int_0^1 \int_0^1 V_{0,3,SU(2)}(t_1, t_2, x) V_{g,n-1,SU(2)}(x, t_3, ..., t_n) dx$$

Proof. We have

$$\lim_{k\to\infty} \frac{1}{k^{3g-3+n}} \#B_{g,n,k}(t_1,...,t_n)$$

$$= \lim_{k\to\infty} \frac{1}{k^2} \sum_{i,j=0}^{k} \#B_{0,3,k}\left(t_1, \frac{i}{k}, \frac{j}{k}\right) \frac{1}{k^{3(g-1)-3+(n+1)}} \#B_{g-1,n+1,k}\left(\frac{i}{k}, \frac{j}{k}, t_2, ..., t_n\right)$$

For $g_1 + g_2 = g$ and $1 + n_1 + n_2 = n$ we have

$$\lim_{k\to\infty} \frac{1}{k^{3g-3+n}} \#B_{g,n,k}(t_1,...,t_n)$$

$$= \lim_{k\to\infty} \frac{1}{k^2} \sum_{i,j=0}^{k} \#B_{0,3,k}\left(t_1, \frac{i}{k}, \frac{j}{k}\right) \frac{1}{k^{3g_1-3+(n_1+1)}} \#B_{g_1,n_1+1,k}\left(\frac{i}{k}, t_2, ..., t_{n_1+1}\right)$$

$$\frac{1}{k^{3g_2-3+(n_2+1)}} \#B_{g_2,n_2+1,k}\left(\frac{j}{k}, t_{n_1+2}, ..., t_n\right)$$

and

$$\lim_{k\to\infty} \frac{1}{k^{3g-3+n}} \#B_{g,n,k}(t_1,...,t_n)$$

$$= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \#B_{0,3,k}\left(t_1, t_2, \frac{i}{k}\right) \frac{1}{k^{3g-3+(n-1)}} \#B_{g,n-1,k}\left(\frac{i}{k}, t_3, ..., t_n\right)$$

Note that $\lim_{k\to\infty} \sum_{i=0}^{k} f(\frac{i}{k}) = \int_0^1 f(x)dx$ for Riemann integrable f. Our functions are Riemann integrable as all our functions are piecewise polynomial which follows from the proof, induction and the fact $V_{0,3}(t_1,t_2,t_3)$ is piecewise polynomial. □

The recursions of corollary 3.1.10 lead to the following formulas for the volume.

Volumes of the Moduli Space		
g	n	$V_{g,n,SU(2)}(t_1,...,t_n)$
0	3	$\begin{cases} 1 & \text{if } \lvert t_1-t_2\rvert \leqslant t_3 \leqslant \min(t_1+t_2, 2-t_1-t_2) \\ 0 & \text{otherwise} \end{cases}$
1	1	$\frac{1}{2}(1-t_1)$
0	4	$\max\left[\frac{1}{2}\min\{\min(t_1+t_2, 2-t_1-t_2), \min(t_3+t_4, 2-t_3-t_4)\} - \frac{1}{2}\max\{\lvert t_1-t_2\rvert, \lvert t_3-t_4\rvert\}, 0\right]$
1	2	$\begin{cases} \frac{1}{4}(1-t_1)t_2 & \text{if } t_1 \geqslant t_2 \\ \frac{1}{4}(1-t_2)t_1 & \text{if } t_1 \leqslant t_2 \end{cases}$
2	1	$\frac{1}{12}t_1(1-t_1)(2-t_1)$
2	2	$\begin{cases} \frac{1}{24}t_1(1-t_1)(2-t_1)t_2 - \frac{1}{24}(1-t_1)t_2^3 & \text{if } t_1 \geqslant t_2 \\ \frac{1}{24}t_2(1-t_2)(2-t_2)t_1 - \frac{1}{24}(1-t_2)t_1^3 & \text{if } t_1 \leqslant t_2 \end{cases}$

Remark: The question is raised as to whether there is a certain transform of the volume such as the Laplace or Fourier transform that simplifies these expressions. This is, in fact, true and is illustrated in Witten's volume formula in the next section.

3.2 The Case of Compact, Connected, Semisimple G

Witten steps through the calculation of the volumes for compact semi-simple connected Lie groups in section 4 of [Wit91]. Considering this more general case, one can see that the structures associated to $SU(2)$ which we proved using the fact that $SU(2)$ has a half dimensional torus action still hold.

3.2.1 Witten's Volume Formula and Intersection Numbers

Notice that in [Wit91] Witten has an additional power of $\frac{1}{2\pi}$ in their formula as they scale their Atiyah-Bott form differently to what we have.

Theorem 3.2.1. *(Witten's Volume Formula)(pg.208 [Wit91])*
Let G be a compact connected semi-simple Lie group. Let $C_1, ..., C_n \in \mathcal{R}_{S^1,G} \cong G/Ad(G)$. Let $irred(G)$ be the irreducible representations of G. Let F be the denominator in the Weyl character formula described in theorem B.2.8. Then we have

$$V_{g,n,G}(C_1, ..., C_n) = \frac{\#Z(G)Vol(G)^{2g-2+n}}{Vol(T)^n} \sum_{\alpha \in irred(G)} \frac{\chi_\alpha(C_1)...\chi_\alpha(C_n)\sqrt{F(C_1)...F(C_n)}}{\dim(\alpha)^{2g-2+n}}$$

Theorem 3.2.2. *(Witten's Volume Formula for $SU(2)$)*

$$V_{g,n,SU(2)}(t_1, ..., t_n) = 2^{5g-4+2n}\pi^{4g-4+n} \sum_{k=1}^{\infty} \frac{\sin(\pi k t_1)... \sin(\pi k t_n)}{k^{2g-2+n}}$$

Theorem 3.2.3. *(Witten's Volume Formula for $SU(2)$ with one Boundary Circle)*
Let P_m be the m-th Bernoulli polynomial.

$$V_{g,1,SU(2)}(t_1) = 2^{5g-2}\pi^{4g-3} \sum_{k=1}^{\infty} \frac{\sin(\pi k t_1)}{k^{2g-1}} = (-2)^g P_{2g-1}\left(\frac{t_1}{2}\right)$$

There are some canonical circle bundles defined on the moduli space of flat connections associated to each boundary component.

Definition: (Canonical Line Bundles on $\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$)
Let $\pi_k : \mathcal{L}_k \to \mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$ be the line bundle where

$$\mathcal{L}_k = \Bigg\{ (A_1, B_1, ..., A_g, B_g, C_1, ..., C_n) \in G^{2g-2+n} : \text{with } Tr(C_i) = 2\cos(\pi t_i)$$

$$A_1 B_1 A_1^{-1} B_1^{-1} ... A_g B_g A_g^{-1} B_g^{-1} C_1, ..., C_n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{with } C_k = \begin{bmatrix} e^{i\pi t_k} & 0 \\ 0 & e^{-i\pi t_k} \end{bmatrix} \Bigg\}$$

and

$$\pi_k : (A_1, B_1, ..., A_g, B_g, C_1, ..., C_n) \mapsto [A_1, B_1, ..., A_g, B_g, C_1, ..., C_n] \in Hom(\pi_1(\Sigma_{g,n}), G)/G$$

There is a result in the theory of symplectic geometry similar to the result of Duistermaat-Heckman. We will simply state the result for $\mathcal{R}_{SU(2),g,n}(t_1, ..., t_n)$.

Theorem 3.2.4. *(Intersection Pairings)(Theorem 4.1 in [Yos01])*
For smooth points $t_1, ..., t_n$ of $V_{g,n,SU(2)}(t_1, ..., t_n)$, which in [Yos01] are referred to as admissible, and sufficiently small $x_1, ..., x_n$ we have

$$V_{g,n,SU(2)}(t_1+x_1, ..., t_n+x_n) = 2^{5g-4+2n}\pi^{4g-4+n}\sum_{k=1}^{\infty}\frac{\sin(\pi k(t_1+x_1))...\sin(\pi k(t_n+x_n))}{k^{2g-2+n}}$$

$$= \sum_{0\leqslant k_1,...,k_n}\frac{x_1^{k_1}}{k_1!}...\frac{x_n^{k_1}}{k_n!}\int_{\mathcal{R}_{SU(2),g,n}(t_1,...,t_n)} c_1(\mathcal{L}_1)^{k_1}...c_1(\mathcal{L}_n)^{k_n}\exp\left(\omega_{g,n,SU(2)}(t_1, ..., t_n)\right)$$

This gives us an explicit way to calculate intersections of cohomological classes. A similar method is used in [Mir06] and [Mir07] to calculate the intersection pairings of the ψ-classes on the moduli space of curves related to 2-dimensional gravity first calculated in [Kon92].

3.2.2 Recursion for Volumes

The recursions of section 3.1.3 generalise to compact connected semi-simple Lie groups as described in Witten's volume formula. Firstly, note the following important cases of Witten's volume formula. Notice that the convergence of these sums to a function is not guaranteed for such large Euler characteristic.

Theorem 3.2.5. *(Volume of the moduli Space for the Cylinder)*
Let G be a compact connected semi-simple Lie group. Let $C_1, C_2 \in \mathcal{R}_{S^1,G} \cong G/Ad(G)$. Let $irred(G)$ be the irreducible representations of G. Let F be the denominator in the Weyl character formula described in theorem B.2.8.

$$V_{0,2,G}(C_1, C_2) = \frac{\#Z(G)}{Vol(T)^2}\sum_{\alpha\in irred(G)}\chi_\alpha(C_1)\chi_\alpha(C_2)\sqrt{F(C_1)F(C_2)}$$

$$= \frac{\#Z(G)}{Vol(T)^2\sqrt{F(C_1)F(C_2)}}\delta_{C_1,C_2}$$

Theorem 3.2.6. *(Volume of the moduli Space for the Pair of Pants)*
Let G be a compact connected semi-simple Lie group. Let $C_1, C_2, C_3 \in \mathcal{R}_{S^1,G} \cong G/Ad(G)$. Let $irred(G)$ be the irreducible representations of G. Let F be the denominator in the Weyl character formula described in theorem B.2.8.

$$V_{0,3,G}(C_1, C_2, C_3) = \frac{\#Z(G)Vol(G)}{Vol(T)^3}\sum_{\alpha\in irred(G)}\frac{\chi_\alpha(C_1)\chi_\alpha(C_2)\chi_\alpha(C_3)\sqrt{F(C_1)F(C_2)F(C_3)}}{\dim(\alpha)}$$

The volumes of surfaces with smaller Euler characteristic can be calculated from surfaces with larger Euler characteristic. Using pair of pants decompositions we can calculate the volume associated to every surface using only the volume of the pair of pants.

Theorem 3.2.7. *(Recursions)*
Let G be a compact connected semi-simple Lie group. Let $C_1, ..., C_n \in \mathcal{R}_{S^1,G} \cong G/Ad(G)$.

$$V_{g,n,G}(C_1, ..., C_n)$$
$$= \frac{Vol(T)^4}{\#Z(G)Vol(G)^2} \int_{G/Ad(G)} \int_{G/Ad(G)} V_{0,3,G}(C_1, A, B)V_{g-1,n+1,G}(A^{-1}, B^{-1}, C_2, ..., C_n)dAdB$$

For $g_1 + g_2 = g$ and $1 + n_1 + n_2 = n$ we have

$$V_{g,n,G}(C_1, ..., C_n) = \frac{Vol(T)^4}{\#Z(G)^2Vol(G)^2} \int_{G/Ad(G)} \int_{G/Ad(G)} V_{0,3,G}(C_1, A^{-1}, B^{-1})$$
$$V_{g_1,n_1+1,G}(A, C_2, ..., C_{n_1+1})V_{g_1,n_1+1,G}(B, C_{n_1+2}, ..., C_n)dAdB$$

and

$$V_{g,n,G}(C_1, ..., C_n)$$
$$= \frac{Vol(T)^2}{\#Z(G)Vol(G)^2} \int_{G/Ad(G)} \int_{G/Ad(G)} V_{0,3,G}(C_1, C_2, A^{-1})V_{g,n-1,G}(A, C_3, ..., C_n)dA$$

Proof. This follows from theorem 3.2.6 and noting that

$$\int_{G/Ad(G)} \chi_\alpha(A)\chi_\beta(A^{-1})F(A)dA = \delta_{\alpha\beta}Vol(G)$$

by the Weyl integration theorem. □

Remark: These kinds of recursions are important in topological recursion. Notice that the volume recursions here are much stronger than the kinds of recursions found there. These recursions in some sense work term by term whereas in topological recursion we need to take a sum over terms like the ones above.

Remark: Notice that the factors in the recursions are closely related to $V_{0,2,G}(C_1, C_2)$. This will later be viewed as part of the gluing associated to some TQFT. To describe this we will abstract a little and show that these volume can be calculated by a TQFT in section 4.2.

Remark: These recursions reflect various properties of the Reidemeister Torsion used by Witten in section 4 of [Wit91] to calculate the volumes.

3.2.3 Comparison to $SU(2)$

Remarkably the formulas of Witten for $SU(2)$ correspond to the piecewise polynomials calculated in sections 3.1.3 and 3.1.2.

Lemma 3.2.8. *For general $t_1, t_2, t_3 \in [0, 1]$*

$$V_{0,3,SU(2)}(t_1, t_2, t_3) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(\pi k t_1)\sin(\pi k t_2)\sin(\pi k t_3)}{k}$$

Proof. To prove this one rewrites $\sin(\pi k t_1) = \frac{e^{i\pi k t_1} - e^{-i\pi k t_1}}{2i}$. Then the summation is of the form

$$\sum_{k=1}^{\infty} \frac{e^{i\pi k(t_1+t_2+t_3)}}{k} + \textit{similar terms}$$

One then using the power series for $\log(1-x)$ which converges on the disc except at 1 which reduces the summation to something of the form

$$\log(1 - e^{i\pi(t_1+t_2+t_3)}) + \textit{similar terms}$$

Then using branching properties of log one can prove the identity. □

This is the base that we build the rest of our volumes from. Therefore to prove that Witten's volume coincides with the volume we calculate all that is left is to check that the recursions are the same.

Lemma 3.2.9. *The recursions determined by Witten's volume formula in theorem 3.2.7 match that determined of lemma 3.1.10.*

Proof. Notice that $SU(2)/Ad(SU(2)) \cong [0,1]$ and that for $A \in SU(2)/Ad(SU(2))$ we have $A^{-1} = A$. This then matches with Witten's volume formula up to some multiplicative factors. □

3.3 Relations to the work of Mirzakhani and Further Directions

3.3.1 The Case of $SL_2(\mathbb{R})$

The case of non-compact G will in general lead to infinite volumes. However, the case where $G = SL_2(\mathbb{R})$ has been extensively studied as the moduli space contains a special connected component called Tiechmüler space. This is the used to define the moduli space of curves. This is of interest in classifying complex curves but moreover making rigorous the concept of 2 dimensional quantum gravity where we want to integrate over the space of metrics. The moduli space of curves gives all equivalence classes of metrics up to isometry.

Definition: (Tiechmüller Space)
Let $L_1, ..., L_n \in \mathbb{R}_{\geqslant 0}$. Then let

$$\mathcal{T}_{g,n}(L_1, ..., L_n)$$

be the set of equivalence classes of hyperbolic metrics on $\Sigma_{g,n}$ with geodesic boundary $L_1, ..., L_n$ with a marking.

Remark: A marking represents a choice of presentation of the fundamental group. The universal cover of a hyperbolic surface with hyperbolic metric defines a representation of the fundamental group into the isometry group of upper half space $PSL_2(\mathbb{R})$. Conjugacy classes in $PSL_2(\mathbb{R})$ define hyperbolic lengths. It can be see that

$$\mathcal{T}_{g,n}(L_1, ..., L_n) \subseteq \mathcal{R}_{PSL_2(\mathbb{R}),g,n}(L_1, ..., L_n)$$

In fact $\mathcal{T}_{g,n}(L_1, ..., L_n)$ is a connected component or the union of two connected components of $\mathcal{R}_{PSL_2(\mathbb{R}),g,n}(L_1, ..., L_n)$.

Definition: (Mapping Class Group)
The mapping class group is defined to be the group of orientation preserving diffeomorphisms modulo isotopies

$$MCG_{g,n} = Diff^+(\Sigma_{g,n})/Iso(\Sigma_{g,n})$$

Remark: This group has a finite presentations in terms of special elements called Dehn twists and half Dehn twists.

Definition: (Moduli Space of Hyperbolic Metrics)
The moduli space of hyperbolic metrics is defined to be

$$\mathcal{M}_{g,n}(L_1, ..., L_n) = \mathcal{T}_{g,n}(L_1, ..., L_n)/MCG_{g,n}$$

Remark: Taking the quotient with respect to the mapping class group gets rid of the marking of the surface.

Remark: In fact $\mathcal{M}_{g,n}(0, ..., 0) = \mathcal{M}_{g,n}$ gives the moduli space of curves. The points of this set determine the isomorphism classes of complex structures on $\Sigma_{g,n}$.

Definition: (Weil-Petterson Volumes)
The symplectic form on $\mathcal{T}_{g,n}$ is invariant under the action of the mapping class group and therefore defines a symplectic form on $\mathcal{M}_{g,n}(L_0, ..., L_n)$ and in particular $\mathcal{M}_{g,n}$ called the Weil-Petterson form. The Weil-Petterson volume is the symplectic volume of $\mathcal{M}_{g,n}$ denoted $V_{PSL_2(\mathbb{R}),g,n}(L_1, ..., L_n)$.

In [Mir06] and [Mir07] Mirzakhani formulates a recursion for the Weil-Petterson volumes. In [Mir07] Mirzakhani points out the analogy between the Weil-Petterson volumes and the volumes of the moduli space of flat connections. Mirzakhani remarks that the mapping class group plays no role in the volumes of the moduli space of flat connections. This makes the recursions for the volume much simpler in this case. Mirzakhani's volume recursions need to take a sum over all pair of pants decompositions whereas the recursions for the volumes of the moduli space of flat connections only needs to take one pair of pants decomposition.

Theorem 3.3.1. *(Mirzakhani's Recursion)([Mir06] [Mir07])*
Let

$$H(x, y) = \frac{1}{1 + e^{\frac{x+y}{2}}} + \frac{1}{1 + e^{\frac{x-y}{2}}}$$

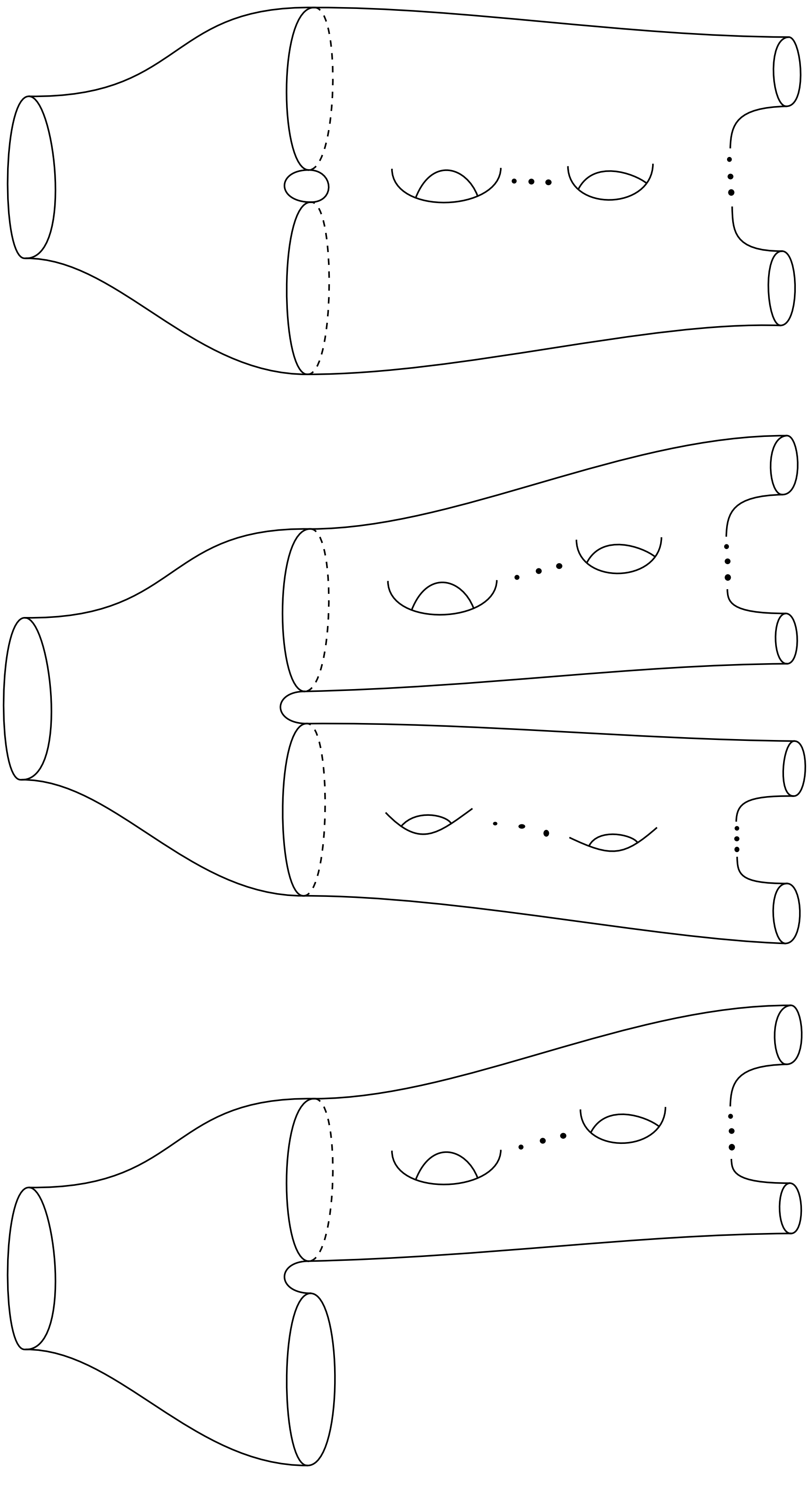
Then we have

$$\frac{\partial}{\partial L_1} L_1 V_{PSL_2(\mathbb{R}),g,n}(L_1, ..., L_n) = \frac{1}{2}\int_0^\infty \int_0^\infty V_{g-1,n+1}(x, y, L_2, ..., L_n)H(x+y, L_1)xydxdy$$

$$+ \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,...,n\}}} \frac{1}{2}\int_0^\infty \int_0^\infty V_{g_1,|I_1|+1}(x, L_{I_1})V_{g_1,|I_2|+1}(y, L_{I_2})H(x + y, L_1)xydxdy$$

$$+ \sum_{i=2}^{n} \frac{1}{2}\int_0^\infty V_{g,n-1}(x, L_2, ..., \widehat{L_i}, ...L_n) \left(H(x, L_1 + L_i) + H(x, L_1 - L_i)\right) xdx$$

The terms in this recursion have use the volumes of the surfaces on the right once we've cut away the pair of pants on the left.



Remark: This is an example of the so-called topological recursion.

3.3.2 Remark on Verlinde's Formula and Recursions for Volume

We will briefly sketch a conjectural relation between Verlinde's dimension formula and the recursions for the volume of the moduli space. To prove such a statement one would have to make more rigorous sense out of the following discussion. For a good starting point see [Tyu03] and [Bea94].

Definition: (WZW CFT)
Given a Lie group G a WZW conformal field theory associates a vector space to every Riemann surface with a labelling at the boundaries. The vector space is defined via sections of a line bundle. To make such a space finite dimensional we want complex structures and to require the sections to be holomorphic.

Consider some surface $\Sigma_{g,n}$. The moduli space of flat connections with boundary holonomies determined by $C_1, ..., C_n \in G/Ad(G)$ denoted as $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$ comes equipped with a symplectic form $\omega_{G,g,n}(C_1, ..., C_n)$. Let P_k be the positive integral weights such that evaluating on the highest root is less than or equal to k. For $C_i \in G/Ad(G)$ satisfying some integrability conditions that embed P_k in $G/Ad(G)$ there exists a line bundle

$$\mathcal{L}_{g,n}(C_1, ..., C_n)^{\otimes k} \to \mathcal{R}_{G,g,n}(C_1, ..., C_n)$$

such that

$$c_1(\mathcal{L}_{g,n}(C_1, ..., C_n)^{\otimes k}) = k\omega_{G,g,n}(C_1, ..., C_n)$$

We can give $\mathcal{R}_{G,g,n}(C_1, ..., C_n)$ a complex structure such that $\mathcal{L}_{g,n}(C_1, ..., C_n)$ is in fact a holomorphic line bundle.

We then take the holomorphic sections of $\mathcal{L}_{g,n}(C_1, ..., C_n)$. This will be a finite dimensional vector space denoted

$$H^0\left(\mathcal{R}_{G,g,n}(C_1, ..., C_n), \mathcal{L}_{g,n}(C_1, ..., C_n)^{\otimes k}\right)$$

Theorem 3.3.2. *(Verlinde's Dimension)(Corollary 9.8 of [Bea94])*
Let P_k be the positive integral weights such that evaluating on the highest root is less than or equal to k. T_k is a finite group analogous to the Weyl group and $h^\vee$ is the dual Coxeter number.

$$D^k_{g,n}(C_1, ..., C_n) = \dim\left(H^0\left(\mathcal{R}_{G,g,n}(C_1, ..., C_n), \mathcal{L}_{g,n}(C_1, ..., C_n)^{\otimes k}\right)\right)$$

$$= |T_k|^{g-1} \sum_{\mu \in P_k} Tr_{V_{C_1,...,C_n}}\left(\exp\left(2\pi i \frac{\mu+\rho}{k+h^\vee}\right)\right) \prod_{\alpha \in R_+} \left|2\sin\left(\pi\frac{\langle\alpha, \mu+\rho\rangle}{k+h^\vee}\right)\right|^{2-2g}$$

See [Bea94] for a proper description of this formula. Note that this is a dimension and so these numbers are integers.

Given a WZW CFT, we can define a (1 + 1)-dimensional topological field theory called the Fusion algebra of the CFT. The Fusion algebra calculates the dimensions of the vector spaces associated to the CFT. The gluing rules in the TQFT (or the Fusion algebra) gives us recursions for the dimensions above. They have the form

$$D^k_{g,n+1}(C_1,...,C_n) = \sum_{x\in P_k} D^k_{0,3}(C_1,C_2,x)D^k_{0,2}(x,y)D^k_{g,n}(y,C_3,...,C_n)$$

We can use the dimensions of the vector space in CFT for surfaces with larger Euler characteristic to determine the Verlinde dimension of surfaces with smaller Euler characteristic. To prove Verlinde's dimension formula we use these recursions which are associated to certain identities in the general theory of Fusion algebras. The following theorem allows us to calculate the volume of the moduli space from these dimensions.

Theorem 3.3.3. *(Riemann-Roch)*
We have the following polynomial in k.

$$\dim\left(H^0\left(\mathcal{R}_{G,g,n}(C_1,...,C_n),\mathcal{L}_{g,n}(C_1,...,C_n)^{\otimes k}\right)\right)$$

$$= \int_{\mathcal{R}_{G,g,n}(C_1,...,C_n)} ch(\mathcal{L}_{g,n}(C_1,...,C_n)^{\otimes k})Td(\mathcal{R}_{G,g,n}(C_1,...,C_n))$$

$$= \int_{\mathcal{R}_{G,g,n}(C_1,...,C_n)} \exp(c_1(\mathcal{L}_{g,n}(C_1,...,C_n)^{\otimes k}))Td(\mathcal{R}_{G,g,n}(C_1,...,C_n))$$

$$= \int_{\mathcal{R}_{G,g,n}(C_1,...,C_n)} \exp(k\omega_{G,g,n}(C_1,...,C_n))Td(\mathcal{R}_{G,g,n}(C_1,...,C_n))$$

$$= k^{\dim_{\mathbb{C}}(\mathcal{R}_{G,g,n}(C_1,...,C_n))}\int_{\mathcal{R}_{G,g,n}(C_1,...,C_n)} \omega_{G,g,n}(C_1,...,C_n)^{\dim_{\mathbb{C}}(\mathcal{R}_{G,g,n}(C_1,...,C_n))}$$

$$+\text{lower order terms in } k$$

where we note that $Td_0(\mathcal{R}_{G,g,n}(C_1,...,C_n)) = 1$. Therefore

$$V_{G,g,n}(C_1,...,C_n) = \int_{\mathcal{R}_{G,g,n}(C_1,...,C_n)} \omega_{G,g,n}(C_1,...,C_n)^{\dim_{\mathbb{C}}(\mathcal{R}_{G,g,n}(C_1,...,C_n))}$$

$$= \lim_{k\to\infty}\frac{1}{k^{\dim_{\mathbb{C}}(\mathcal{R}_{G,g,n}(C_1,...,C_n))}}\dim\left(H^0\left(\mathcal{R}_{G,g,n}(C_1,...,C_n),\mathcal{L}_{g,n}(C_1,...,C_n)^{\otimes k}\right)\right)$$

Conjecture 3.3.4. *(Recursions for the Volume from Verlinde's Formula)*
Conjecturally in the limit the contractions associated to gluing in the fusion algebra become integration over the conjugacy classes. Thereby recovering the recursions of theorem 3.2.7.

$$\frac{1}{k^{rank(G)}}\sum_{x\in P_k} \to \int_{G/Ad(G)}$$

Remark: In the $SU(2)$ case we saw

$$V_{g,n}(t_1,...,t_n) = \lim_{k\to\infty} \frac{1}{k^{3g-3+n}} \# B_{g,n,k}(t_1,...,t_n)$$

$$= \lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \# B_{0,3,k}\left(t_1, t_2, \frac{i}{k}\right) \frac{1}{k^{3g-3+(n-1)}} \# B_{g,n-1,k}\left(\frac{i}{k}, t_3, ..., t_n\right)$$

$$= \int_0^1 V_{0,3}(t_1,t_2,x) V_{g,n-1}(x,t_3,...,t_n) dx$$

Remark: In [ABO15] and [MOP+13] there is a so-called cohomological field theory whose topological field theory is given by the fusion algebra of a WZW CFT. It may be interesting whether one could make sense of this limiting procedure when considering the cohomological field theories and not just the topological field theories.

Remark: We have a $\mathbb{Z}_{>0}$ set of $(1+1)$-dimensional TQFTs that in some limit calculate some volume. The next section gives us a way to make this volume into a TQFT of its own.

Remark: Also notice that Riemann-Roch and the calculations of section 3.1.2 give the following kinds of expressions.

$$\int_{\mathcal{R}_{SU(2),1,2}(t_1,t_2)} \omega_{SU(2),g,n}(t_1,t_2) Td_1(\mathcal{R}_{SU(2),g,n}(t_1,t_2))$$

$$= \left(\frac{1}{2}\min(t_1+t_2, 2-t_1-t_2) - \frac{3}{2}|t_1-t_2| + 1\right)$$

These expressions don't immediately follow from Verlinde's formula and it would be interesting to see if they have nice general expressions similar to the volume of the moduli space.

If this could be done, Verlinde's formula could be simplified in the sense that one wouldn't have to sum over P_k and it would be represented by a polynomial in k with functions as coefficients.

Chapter 4

Volumes of the Moduli Space as a Topological Quantum Field Theory

TQFTs can naturally count locally defined structures. This gives an interesting approach to enumerative geometry. In the first part of this section we consider the extremely concrete example of Dijkgraaf-Witten TQFT and then notice a remarkable relation to the volumes moduli space of flat connections.

4.1 Trivial Dijkgraaf-Witten Topological Quantum Field Theory

From a physical perspective finite gauge groups are unusual. However, from a mathematical perspective, they allow for a well defined TQFT that calculates the numbers of representations of the fundamental group into a finite group. This was originally described in [DW90] and a nice presentation can also be found in [FQ93]. The following sections describe a trivial version of these theories in the sense that the action functional is trivial.

4.1.1 Finite Gauge Groups and Covering Spaces

As mentioned in Appendix C for a finite group G we can define a principle G-bundle.

Definition: (G-covers of M)
Let G be a finite group with discrete topology and $(\pi : P \to M, F)$ a smooth fibre bundle. If G acts smoothly on P via $\cdot : P \times G \to P$ such that for $p \in P$ and $g \in G$

$$\pi(p \cdot g) = \pi(p)$$

and G acts freely and transitively on $\pi^{-1}(x)$ for all $x \in M$ then we say $(\pi : P \to M, F, G, \cdot)$ is a **G-cover of** M. Notice that $M \cong P/G$ and $F \cong G$. So we can write our G-cover of M as $(\pi : P \to M, G, \cdot)$.

Remark: This defines a covering space with Deck transformations given by G.

Definition: (Pull Back and Push Forward G-Covers)
Let N be a smooth manifold, $(\pi : P \to M, G, \cdot)$ a G-cover and let $f : N \to M$ be a smooth map. We define the **pull back G-cover** induced by f as follows

$$f^*P = \{(x,p) \in N \times P : f(x) = \pi(p)\}$$

If f is a diffeomorphism define

$$(f^{-1})_*P = f^*P$$

These have the natural structure of a G-cover.

The following theorem is analogous to theorem 1.1.8.

Theorem 4.1.1. *Let M be a connected smooth manifold and G a finite group with the discrete topology. Let $\mathcal{R}_{M,G}$ be the isomorphism classes of G-covers of M. Then there is a bijection*

$$\mathcal{R}_{M,G} \cong Hom(\pi_1(M), G)/G$$

Proof. See the proof of theorem 1.1.8. For a feeling of what is going on notice that the universal cover of M is unique up to unique isomorphism with Deck transformations given by $\pi_1(M)$. Every cover must factor through the universal cover which gives rise to the representation classes of the fundamental group. □

Remark: Theorem 4.1.1 is one of the reasons to consider the flat connections when we let G be a Lie group. Given a Lie group G the flat connections correspond to G-covers of the base where we instead give G the discrete topology. These spaces will in general be slightly pathological as the $M \times G$ looks like it has dimension $\dim(M) + \dim(G)$ but giving G the discrete topology means that we are viewing $\dim(M \times G) = \dim(M)$.

Remark: For smooth connected compact manifolds the fundamental group has a finite presentation. So $\mathcal{R}_{M,G}$ will be a finite set for a finite group G.

Definition: (Automorphisms of G-cover)
Let $(\pi : P \to M, G, \cdot)$ be a G-cover. Let the group of automorphisms of P be denoted

$$Aut(P) = \{A : P \to P : A \text{ is an equivariant bundle map covering the identity}\}$$

For $(\pi : P \to M, G, \cdot)$ and $p \in \pi^{-1}(x)$ we can define a representation of the fundamental group $\rho_p \in Hom(\pi_1(M,x), G)$ (see corollary E.2.2 and adapt it to G-covers). The based group of automorphisms at p is defined to be $Aut_p(P) = \{g \in G : g\rho_p g^{-1} = \rho_p\}$.

Remark: Automorphisms of G-covers correspond to the gauge transformations in section 1.1.2.

Remark: Notice that $Aut_p(A) \cong Stab(\rho_p)$ with respect to the action of G acting on $Hom(\pi_1(M, x), G)$ by conjugation.

Lemma 4.1.2. *Let M be a connected smooth manifold and $(\pi : P \to M, G, \cdot)$ be a G-cover. We have the following identification*

$$Aut(P) \cong Aut_p(P)$$

Proof. Firstly notice that for $F : P \to P \in Aut(P)$ we can define a smooth map $f : P \to G$ with $F(p) = p \cdot f(p)$ as in section 1.1.2. f is equivariant in the sense that $f(p \cdot g) = g^{-1} f(p) g$. For $f : P \to G$ to be smooth f must be constant on the connected components of P.

Every connected component of P intersects the fibre $\pi^{-1}(x)$ as M is connected. This means that f is completely determined by it restriction to the fibre $f|_{\pi^{-1}(x)}$. However the f restricted to the fibre is determined by $f(p)$ as $f(p \cdot g) = g^{-1} f(p) g$. This means that the function f is determined by the value $f(p)$.

Now $f(p)$ can't be any element of G as for $p \cdot g$ in the same connected component of P we must have $f(p) = f(p \cdot g)$. Notice that if $p \cdot g$ is in the same path component as p then by definition there is some path $\tilde{\gamma} : [0, 1] \to P$ such that $\tilde{\gamma}(0) = P$ and $\tilde{\gamma}(1) = p \cdot g$. This means that the lift of $\pi \circ \tilde{\gamma}$ is given by $\tilde{\gamma}$ and therefore $\rho_p([\pi \circ \tilde{\gamma}]) = g$. So $p \cdot g$ is in the same path component as p if and only if there is some $[\gamma] \in \pi_1(M, x)$ such that $\rho_p([\gamma]) = g$.

Therefore the only condition on $f(p)$ is that $f(p \cdot \rho_p([\gamma])) = \rho_p([\gamma])^{-1} f(p) \rho_p([\gamma]) = f(p)$. Restating this as $\rho_p([\gamma]) = f(p) \rho_p([\gamma]) f(p)^{-1}$ for all $[\gamma] \in \pi_1(M, x)$ we see that f is determined by and determines an element of G given by $f(p)$ such that $\rho_p = f(p) \rho_p f(p)^{-1}$. Therefore we have $Aut(P) \cong Aut_p(P)$. $\square$

4.1.2 Categorification to a Topological Quantum Field Theory

The equivalence of G-covers defined in the last section is given by bundle automorphisms which cover the identity. This equivalence is therefore local with respect to the base of the cover. Gluing automorphism classes of G-covers can then be done in a consistent way. This means, that by considering G-covers on some smaller manifolds and then gluing them together, we should be able to build up the more complex examples. For example, to count the number of G-covers of a given manifold, we can use recursions similar to the volume recursions of section 3. The question is raised as to whether there is a TQFT that encodes these recursions.

A naïve guess at a TQFT would be to take the vector space associated to a d-dimensional manifold Σ to be $Z(\Sigma) = \{f : \mathcal{R}_{\Sigma,G} \to \mathbb{C}\} = Span_{\mathbb{C}}\{\mathcal{R}_{\Sigma,G}\}$ and the functional associated to a $(d+1)$-dimensional manifold M to be defined on the canonical basis of $Z(\partial M) = Span_{\mathbb{C}}\{\mathcal{R}_{\partial M,G}\}$ such that for $A \in \mathcal{R}_{\partial M,G}$ we have $Z(M, f, \emptyset)(A) = \#\{B \in \mathcal{R}_{M,G} : B|_{\partial M} = f_*(A)\}$.

However, this will not work as the different ways of gluing the covers together is not taken into account. The different ways of gluing the covers together is determined by the automorphisms of the cover restricted to boundary being glued together. This motivates the following definition/theorem. However, first we have the following notation.

Notation: Let $A \in \mathcal{R}_{\partial M,G}$. Let $\mathcal{R}_{M,G}(A) = \{B \in \mathcal{R}_{M,G} : B|_{\partial M} = A\}$.

Theorem 4.1.3. *(Dijkgraaf-Witten TQFT)*
Let G be a finite group. The following data uniquely determines a $(d+1)$-dimensional TQFT over $\mathbb{C}$ denoted Z_G such that for a d-dimensional manifold Σ and $(d+1)$-dimensional manifold M.

- $Z_G(\Sigma) = Span_{\mathbb{C}}\{\mathcal{R}_{\Sigma,G}\}$
- $Z_G(M, f, \emptyset) : Z_G(\partial M) \to \mathbb{C}$ *is the linear map such that for* $A \in \mathcal{R}_{\partial M,G}$ *we have* $Z_G(M, f, \emptyset)(A) = \sum_{B \in \mathcal{R}_{M,G}(f_*(A))} \frac{1}{\#Aut(B)}$

To prove this statement requires an understanding of how this information allows gluing of cobordisms. So far we only have ingoing boundary components. The key is to use the identity axiom for TQFTs exploited in the following lemma.

Lemma 4.1.4. *(Gluing in Z_G)*
For the d-dimensional TQFT Z_G and $A_1, A_2 \in \mathcal{R}_{\Sigma,G}$ we have

$$Z_G(\Sigma \times [0,1], id_{\Sigma*} \sqcup id_{\Sigma*}, \emptyset)(A_1, A_2) = \delta_{A_1,A_2}\frac{1}{\#Aut(A_1)} = \delta_{A_1,A_2}\frac{1}{\#Aut(A_2)}$$

Therefore

$$Z_G\left(\Sigma \times [0,1], \emptyset, id_\Sigma \sqcup id_\Sigma\right) = \sum_{A \in \mathcal{R}_{\Sigma,G}} \#Aut(A) A \otimes A$$

Proof. Noting theorem 4.1.1 and that we have a homotopy $\Sigma \times [0,1] \simeq \Sigma$ we see that any G-cover $\pi : P \to \Sigma \times [0,1]$ we must have $P|_{\Sigma\times\{0\}} \cong P|_{\Sigma\times\{1\}}$. Noting lemma 4.1.2 we then see that $Aut(P) \cong Aut(P|_{\Sigma\times\{0\}})$. Therefore we have

$$Z_G(\Sigma \times [0,1], id_{\Sigma*} \sqcup id_{\Sigma*}, \emptyset)(A_1, A_2) = \delta_{A_1,A_2}\frac{1}{\#Aut(A_1)} = \delta_{A_1,A_2}\frac{1}{\#Aut(A_2)}$$

Let

$$Z_G\left(\Sigma \times [0,1], \emptyset, id_\Sigma \sqcup id_\Sigma\right) = \sum_{A_1, A_2 \in \mathcal{R}_{\Sigma,G}} \alpha_{A_1,A_2} A_1 \otimes A_2$$

Consider the following pictorial representations.

$$(\Sigma \times [0,1], id_{\Sigma*} \sqcup id_{\Sigma*}, \varnothing) \qquad = \qquad \begin{matrix} id_{\Sigma*} \\ \Big) \\ id_{\Sigma*} \end{matrix}$$

$$(\Sigma \times [0,1], \varnothing, id_{\Sigma} \sqcup id_{\Sigma}) \qquad = \qquad \begin{matrix} id_{\Sigma} \\ \Big(\\ id_{\Sigma} \end{matrix}$$

From the TQFT axioms and the following pictures we have

$$\begin{matrix} & id_{\Sigma*} \\ & \big) \\ id_{\Sigma} & - \ id_{\Sigma*} \\ \big(& \\ id_{\Sigma} & \end{matrix} \qquad = \qquad id_{\Sigma*} = id_{\Sigma}$$

$$\sum_{A_2 \in \mathcal{R}_{\Sigma,G}} \alpha_{B,A_2} \frac{1}{Aut(B)} A_2 = \sum_{A_1, A_2 \in \mathcal{R}_{\Sigma,G}} \alpha_{A_1,A_2} \frac{\delta_{B,A_1}}{Aut(B)} A_2$$

$$= \sum_{A_1, A_2 \in \mathcal{R}_{\Sigma,G}} \alpha_{A_1,A_2} Z_G(\Sigma \times [0,1], id_{\Sigma*} \sqcup id_{\Sigma*}, \varnothing)(B, A_1) A_2 = id_{Z_G(\Sigma)}(B) = B$$

This means that $\alpha_{B,A_2} = \delta_{B,A_2} Aut(B)$. Therefore

$$Z_G\left(\Sigma \times [0,1], \varnothing, id_{\Sigma} \sqcup id_{\Sigma}\right) = \sum_{A \in \mathcal{R}_{\partial M,G}} \#Aut(A) A \otimes A$$

$\square$

Remark: Using $Z_G\left(\Sigma \times [0,1], \varnothing, id_{\Sigma} \sqcup id_{\Sigma}\right)$ we can determine Z_G given the information of theorem 4.1.3 as follows.

Definition: $(Z_G : \underline{Cob_{d+1}} \to \underline{Vec_{\mathbb{C}}})$

• Let $Z_G : ob\left(\underline{Cob_{d+1}}\right) \to ob\left(\underline{Vec_{\mathbb{C}}}\right)$ such that $Z_G(\Sigma) = Span_{\mathbb{C}}\{\mathcal{R}_{\Sigma,G}\}$ and $Z_G(\varnothing) = \mathbb{C}$.

• For $(M, f_1, f_2) \in Hom_{\underline{Cob_{d+1}}}(\Sigma_1, \Sigma_2)$ and for $A \in \mathcal{R}_{\Sigma_1,G}$ viewed as $A \in Z_G(\Sigma_1) = Span_{\mathbb{C}}\{\mathcal{R}_{\Sigma_1,G}\}$ let

$$Z_G(M, f_1, f_2)(A) = \sum_{C \in \mathcal{R}_{\Sigma_2,G}} \left(\sum_{B \in \mathcal{R}_{M,G}((f_1)_*(A) \sqcup (f_2)_*(C))} \frac{\#Aut(C)}{\#Aut(B)} \right) C$$

Corollary 4.1.5. *(Z_G is a TQFT except for the Gluing Axiom)*

- *For* $\Sigma \in ob(\underline{Cob_{d+1}})$ *we have* $Z_G(\Sigma^*) = Z_G(\Sigma)^*$.
- *For* $\Sigma_1, \Sigma_2 \in ob(\underline{Cob_{d+1}})$ *we have* $Z_G(\Sigma_1 \sqcup \Sigma_2) = Z_G(\Sigma_1) \otimes Z(\Sigma_2)$.
- *For* $\varnothing \in ob(\underline{Cob_{d+1}})$ *we have* $Z(\varnothing) = \mathbb{C}$.
- *For* $\Sigma \in ob(\underline{Cob_{d+1}})$ *and* $(\Sigma \times [0,1], Id_{\Sigma *}, Id_{\Sigma}) \in Hom_{\underline{Cob_{d+1}}}(\Sigma, \Sigma)$ *we have* $Z(\Sigma \times [0,1], Id_{\Sigma *}, Id_{\Sigma}) = id_{Z(\Sigma)} \in Hom_{\underline{Cob_{d+1}}}(Z(\Sigma), Z(\Sigma))$.

The only thing left to check is that Z_G satisfies the gluing axiom. To prove this, we need the following lemmas. We will suppress some of the notation in the next few lemmas. In particular, we will not name all the inclusions of the boundary and certain isomorphisms.

Definition: (Gluing Covers Together)
Let Σ be a smooth closed d-manifold, M_1, M_2 smooth $(d+1)$ manifolds and let $(\pi_1 : P_1 \to M_1, G, \cdot)$ and $(\pi_2 : P_2 \to M_2, G, \cdot)$ be G-covers with a fixed inclusion $\partial M_1 \supseteq \Sigma \subseteq \partial M_2$ such that for some $C \in \mathcal{R}_{G,\Sigma}$ we have $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$ with these isomorphism covering the fixed inclusions and fix these isomorphisms. For $\varphi \in Aut(C)$ define

$$P_1 \cup_C^{\varphi} P_2 = P_1 \sqcup P_2 / \sim$$

to be the space where $p_1 \sim p_2$ if $p_1 \in \pi_1^{-1}(\Sigma) \subseteq P_1$ and $p_2 \in \pi_2^{-1}(\Sigma) \subseteq P_2$ and using the fixed isomorphisms $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$ we have $p_1 = \varphi(p_2)$.

Let $\pi : P_1 \cup_C^{\varphi} P_2 \to M_2 \cup_{\Sigma} M_2$ such that for $p_1 \in P_1$ and $p_2 \in P_2$ we have $\pi(p_1) = \pi_1(p_1)$ and $\pi(p_2) = \pi_2(p_2)$. This is well defined as using the fixed isomorphisms $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$ we have $\pi_1|_C = \pi_2|_C$ and so $\pi_1|_C(p) = \pi_2|_C(p) = \pi_2|_C(\varphi(p))$.

We then have an action of G on $P_1 \cup_C^{\varphi} P_2$ given by the G action on P_1 and P_2. Notice that this is well defined as $p \cdot g \sim \varphi(p \cdot g) = \varphi(p) \cdot g$.

Note that in particular φ is a diffeomorphism so we can give $P_1 \cup_C^{\varphi} P_2$ a smooth structure induced from the gluing. Taking local trivialisations at points in C in P_1 and P_2 we can see that the $P_1 \cup_C^{\varphi} P_2$ is locally trivial and therefore we have a G-cover defined by $(\pi : P_1 \cup_C^{\varphi} P_2 \to M_1 \cup_{\Sigma} M_2, G, \cdot)$.

Lemma 4.1.6. *Let Σ be a smooth closed d-manifold, M a connected smooth $(d+1)$ manifold and let $(\pi : P \to M, G, \cdot)$ be a G-cover with a fixed inclusion $\Sigma \subseteq \partial M$ such that $\pi^{-1}(\Sigma) \cong C$ covering the inclusion and fix this isomorphism.*

Then $Aut(P) \leqslant Aut(C)$.

Proof. Let $r : Aut(P) \to Aut(C)$ such that $r(\varphi) = \varphi|_C$. We claim this is an injection. Notice from the proof of lemma 4.1.2 that the automorphisms of P are determined by a single element of G. Therefore if $r(\varphi_1) = r(\varphi_2)$ they must extend to the same unique automorphism of P. In other words $\varphi_1 = \varphi_2$. □

To prove Z_G is a TQFT, we need to understand the number of automorphisms of a G-cover glued together along some boundary. This uses the following lemmas.

Lemma 4.1.7. *Let Σ be a smooth closed d-manifold, M_1, M_2 be connected smooth $(d+1)$ manifolds and let $(\pi_1 : P_1 \to M_1, G, \cdot)$ and $(\pi_2 : P_2 \to M_2, G, \cdot)$ be G-covers with a fixed inclusions $\partial M_1 \supseteq \Sigma \subseteq \partial M_2$ such that $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$ covering the inclusions and fix these isomorphisms. We have*

$$P_1 \cup_C^{\varphi} P_2 \cong P_1 \cup_C^{\psi} P_2$$

if and only if there exists $\alpha_1 \in Aut(P_1) \leqslant Aut(C)$, $\alpha_2 \in Aut(P_2) \leqslant Aut(C)$ such that

$$\alpha_2 \circ \varphi = \psi \circ \alpha_1$$

Proof. "$\Leftarrow$"
If $\alpha_2 \circ \varphi = \psi \circ \alpha_1$. Then take $F : P_1 \cup_C^{\varphi} P_2 \cong P_1 \cup_C^{\psi} P_2$ such that

$$F(p) = \begin{cases} \alpha_1(p) & \text{if } p \in P_1 \\ \alpha_2(p) & \text{if } p \in P_2 \end{cases}$$

This is well defined as for $p \in C$ we have $p \sim_{\varphi} \varphi(p)$ we have $F(p) = \alpha_1(p) \sim_{\psi} \psi \circ \alpha_1(p)$ and $F(\varphi(p)) = \alpha_2 \circ \varphi(p) = \psi \circ \alpha_1(p) = \psi \circ \alpha_1(p) \sim_{\psi} \alpha_1(p)$.

Notice that we have inverse $F^{-1} : P_1 \cup_C^{\psi} P_2 \cong P_1 \cup_C^{\varphi} P_2$ such that

$$F^{-1}(p) = \begin{cases} \alpha_1^{-1}(p) & \text{if } p \in P_1 \\ \alpha_2^{-1}(p) & \text{if } p \in P_2 \end{cases}$$

This is well defined as for $p \in C$ we have $p \sim_{\psi} \psi(p)$ we have $F^{-1}(p) = \alpha_1^{-1}(p) \sim_{\varphi} \varphi \circ \alpha_1^{-1}(p)$ and $F^{-1}(\psi(p)) = \alpha_2^{-1} \circ \psi(p) = \varphi \circ \alpha_1^{-1}(p) \sim_{\varphi} \alpha_1^{-1}(p)$.

"$\Rightarrow$"
If $F : P_1 \cup_C^{\varphi} P_2 \cong P_1 \cup_C^{\psi} P_2$ then let $F|_{P_1} = \alpha_1 \in Aut(P_1)$ and $F|_{P_2} = \alpha_2 \in Aut(P_2)$. For $p \in C$ we have $F(p) \sim_{\psi} F \circ \varphi(p)$ and so $\alpha_1(p) \sim_{\psi} \alpha_2 \circ \varphi(p)$ however we also have $\alpha_1(p) \sim_{\psi} \psi \circ \alpha_1(p)$. So $\alpha_2 \circ \varphi(p) = \psi \circ \alpha_1(p)$. □

Corollary 4.1.8. *Let Σ be a smooth closed d-manifold, M_1, M_2 be connected smooth $(d+1)$ manifolds and let $(\pi_1 : P_1 \to M_1, G, \cdot)$ and $(\pi_2 : P_2 \to M_2, G, \cdot)$ be G-covers with a fixed inclusions $\partial M_1 \supseteq \Sigma \subseteq \partial M_2$ such that $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$ covering the inclusions and fix these isomorphisms. Then for $\varphi \in Aut(C)$ we have $F : P_1 \cup_C^{\varphi} P_2 \cong P_1 \cup_C^{\varphi} P_2$ if and only if there exists $\alpha_1 \in Aut(P_1)$, $\alpha_2 \in Aut(P_2)$ such that $F|_{P_1} = \alpha_1$ and $F|_{P_2} = \alpha_2$ and*

$$\alpha_2 \circ \varphi = \varphi \circ \alpha_1$$

Proof. Apply the proof of 4.1.7 for $\psi = \varphi$. □

Corollary 4.1.9. *Let Σ be a smooth closed d-manifold, M_1, M_2 be connected smooth $(d+1)$ manifolds with fixed inclusions $\partial M_1 \supseteq \Sigma \subseteq \partial M_2$ and let $M_1 \cup_\Sigma M_2 = M$. Let $(\pi : P \to M, G, \cdot)$ be a G-cover, $P|_{M_1} = (\pi_1 : P_1 \to M_1, G, \cdot)$, $P|_{M_2} = (\pi_2 : P_2 \to M_2, G, \cdot)$ and $P|_\Sigma = C$. Then*

$$\#\{\varphi \in Aut(C) : P \cong P_1 \cup_C^\varphi P_2\} = \frac{\#Aut(P_1)\#Aut(P_2)}{\#Aut(P)}$$

Proof. P induces fixed isomorphisms $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$. So there exists $\psi \in Aut(C)$ such that $P = P_1 \cup_C^\psi P_2$ by simply restricting P to M_1 and M_2 and taking note of the fixed isomorphisms $\pi_1^{-1}(\Sigma) \cong C \cong \pi_2^{-1}(\Sigma)$.

For each $\alpha_1 \in Aut(P_1)$ and $\alpha_2 \in Aut(P_2)$ we get $\varphi = \alpha_2^{-1} \circ \psi \circ \alpha_1$ such that $P \cong P_1 \cup_C^\varphi P_2$ by lemma 4.1.7 and this gives all possible φ again by lemma 4.1.7. However by the corollary 4.1.8 we have over counted each φ by a factor of $\#Aut(P) = \#Aut(P_1 \cup_C^\varphi P_2)$. □

Lemma 4.1.10. *(Automorphisms of Glued Covers)*
Let G a finite group, M_1 and M_2 connected smooth manifolds with fixed inclusions $\partial M_1 \supseteq \Sigma \subseteq \partial M_2$ and $M = M_1 \cup_\Sigma M_2$. Let $B \in \mathcal{R}_{\partial M,G}$ with $B_1 = B|_{\partial M_1 - \Sigma}$ and $B_2 = B|_{\partial M_2 - \Sigma}$. For $C \in \mathcal{R}_{\Sigma,G}$ let $\mathcal{S}(C) = \{(P_1, P_2) \in \mathcal{R}_{M_1,G}(B_1, C) \times \mathcal{R}_{M_2,G}(C, B_2)\}$. Then

$$\sum_{P \in \mathcal{R}_{M,G}(B)} \frac{1}{\#Aut(P)} = \sum_{C \in \mathcal{R}_{\Sigma,G}} \left(\sum_{(P_1,P_2) \in \mathcal{S}(C)} \frac{\#Aut(C)}{\#Aut(P_1)\#Aut(P_2)} \right)$$

Proof. By corollary 4.1.9 we have

$$\frac{\#\{\varphi \in Aut(C) : P \cong P|_{M_1} \cup_C^\varphi P|_{M_2}\}}{\#Aut(P|_{M_1})\#Aut(P|_{M_2})} = \frac{1}{\#Aut(P)}$$

Notice that if we take all pairs $(P_1, P_2) \in \mathcal{S}(C)$ and all $\varphi \in Aut(C)$ we will get all possible bundles P such that $P|_\Sigma = C$. We will however over count a given bundle P by a factor of $\#\{\varphi \in Aut(C) : P \cong P|_{M_1} \cup_C^\varphi P|_{M_2}\}$. This means that

$$\sum_{(P_1,P_2) \in \mathcal{S}(C)} \frac{\#Aut(C)}{\#Aut(P_1)\#Aut(P_2)} = \sum_{(P_1,P_2) \in \mathcal{S}(C)} \left(\sum_{\varphi \in Aut(C)} \frac{1}{\#Aut(P_1)\#Aut(P_2)} \right)$$

$$= \sum_{\substack{P \in \mathcal{R}_{M,G}(B) \\ s.t\ P|_\Sigma = C}} \frac{\#\{\varphi \in Aut(C) : P \cong P|_{M_1} \cup_C^\varphi P|_{M_2}\}}{\#Aut(P|_{M_1})\#Aut(P|_{M_2})} = \sum_{\substack{P \in \mathcal{R}_{M,G}(B) \\ s.t\ P|_\Sigma = C}} \frac{1}{\#Aut(P)}$$

Then summing over $C \in \mathcal{R}_{\Sigma,G}$ gives

$$\sum_{P \in \mathcal{R}_{M,G}(B)} \frac{1}{\#Aut(P)} = \sum_{C \in \mathcal{R}_{\Sigma,G}} \left(\sum_{\substack{P \in \mathcal{R}_{M,G}(B) \\ s.t\ P|_\Sigma = C}} \frac{1}{\#Aut(P)} \right)$$

$$= \sum_{C \in \mathcal{R}_{\Sigma,G}} \left(\sum_{(P_1,P_2) \in \mathcal{S}(C)} \frac{\#Aut(C)}{\#Aut(P_1)\#Aut(P_2)} \right)$$

□

Corollary 4.1.11. *(Theorem 4.1.3)*
Z_G is a TQFT.

Proof. We checked everything except the composition or gluing axiom in corollary 4.1.5.

When gluing disconnected manifolds we can simply glue a connected component at a time and therefore reduce the proof that the gluing axiom is satisfied to showing it is satisfied when gluing connected manifolds.

We used the calculation in lemma 4.1.4 to define Z_G. Now to make sure that this is consistent and the gluing axiom holds take connected smooth manifolds M_1 and M_2 with inclusions $\iota_1 : \Sigma \hookrightarrow \partial M_1$, $\iota_2 : \Sigma \hookrightarrow \partial M_2$, $f_1 : \Sigma_1 \hookrightarrow \partial M_1 - \iota_1(\Sigma)$ and $f_2 : \Sigma_2 \hookrightarrow \partial M_2 - \iota_2(\Sigma)$. Let $M = M_1 \cup_{\Sigma}^{\iota_1,\iota_2} M_2$. We need to show that for $(M_1, f_1 \sqcup \iota_1, \varnothing)$, $(M_2, \iota_2 \sqcup f_2, \varnothing)$, $(M, f_1 \sqcup f_2, \varnothing)$ and $B \in \mathcal{R}_{\Sigma_1 \sqcup \Sigma_2, G}$ with $B|_{\Sigma_1} = B_1 \in \mathcal{R}_{\Sigma_1,G}$ and $B|_{\Sigma_2} = B_2 \in \mathcal{R}_{\Sigma_2,G}$ we have

$$Z_G(M, f_1 \sqcup f_2, \varnothing)(B)$$

$$= \sum_{C \in \mathcal{R}_{\Sigma,G}} \#Aut(C) Z_G(M_1, f_1 \sqcup \iota_1, \varnothing)(B_1, C) Z_G(M_2, \iota_2 \sqcup f_2, \varnothing)(C, B_2)$$

This is exactly the content of lemma 4.1.10. □

The invariants associated to Z_G can be explicitly calculated using the fundamental group. Notice that in particular this means that these invariants are therefore weaker than the fundamental group. We have the following result for general dimension.

Theorem 4.1.12. *(Partition Function for Dijkgraaf-Witten TQFT)*
Let M be a connected smooth manifold, $\partial M \cong \Sigma_1 \sqcup ... \sqcup \Sigma_n$ with Σ_i connected and G a finite group. Let Z_G be the Dijkgraaf-Witten TQFT of the dimension of M. Then for $A \in \mathcal{R}_{\Sigma_1 \sqcup ... \sqcup \Sigma_n, G}$ and $f_i : \Sigma_i \hookrightarrow \partial M$ with corresponding representation for $(f_i)_*(A|_{\Sigma_i})$ given by $\rho_{A_i} \in Hom(\pi_1(f_i(\Sigma_i)), G)/G$ we have

$$Z_G(M, f_1 \sqcup ... \sqcup f_n, \varnothing)(A) = \frac{\#\{\rho \in Hom(\pi_1(M), G) : [\rho|_{\pi_1(f_i(\Sigma_i))}] = \rho_{A_i}\}}{\#G}$$

Proof. By the definition of Z_G for $A \in \mathcal{R}_{\partial M,G} \subseteq Z_G(\partial M)$ we have

$$Z_G(M, f_1 \sqcup ... \sqcup f_n, \varnothing)(A) = \sum_{B \in \mathcal{R}_{M,G}(f_*(A))} \frac{1}{\#Aut(B)}$$

Notice that by by lemma 4.1.2 and the proceeding remark we see that if ρ_B represents the representation class associated to the bundle B we have $\#Aut(B) = \#Stab(\rho_B)$.

Then by theorem 4.1.1 we have

$$\sum_{B\in\mathcal{R}_{M,G}(f_*(A))}\frac{1}{\#Aut(B)}=\sum_{\substack{\rho\in Hom(\pi_1(M),G)/G\\ s.t\quad [\rho|_{\Sigma_i}]=\rho_{A_i}}}\frac{1}{\#Stab(\rho)}$$

Now by the orbit stabiliser relation we have

$$\frac{1}{\#Stab(\rho)}=\frac{\#Orb(\rho)}{\#G}$$

Combining this with the previous statements gives

$$Z_G(M)(A)=\sum_{B\in\mathcal{R}_{M,G}(f_*(A))}\frac{1}{\#Aut(B)}=\sum_{\substack{\rho\in Hom(\pi_1(M),G)/G\\ s.t\quad [\rho|_{\pi_1(f_i(\Sigma_i))}]=\rho_{A_i}}}\frac{\#Orb(\rho)}{\#G}$$

$$=\sum_{\substack{\rho\in Hom(\pi_1(M),G)\\ s.t\quad [\rho|_{\pi_1(f_i(\Sigma_i))}]=\rho_{A_i}}}\frac{1}{\#G}=\frac{\#\{\rho\in Hom(\pi_1(M),G):[\rho|_{\pi_1(f_i(\Sigma_i))}]=\rho_{A_i}\}}{\#G}$$

□

The calculation can be taken even further in the case of $(1+1)$-dimensional TQFT. This exploits the classification of $(1+1)$-dimensional TQFTs discussed in section G.3 to get the following results.

Theorem 4.1.13. *(Partition Function for Dijkgraaf-Witten TQFT on a Surface) Let $\Sigma_{g,n}$ be the closed surface of genus g with n boundary circles given by $C_1,...,C_n\cong S^1$, G a finite group, $irred(G)$ the finite dimensional irreducible $\mathbb{C}$-representations of G and $\chi_\alpha: G\to\mathbb{C}$ the character of the representation α. Consider Z_G the Dijkgraaf-Witten TQFT at dimension $1+1$. For $A_1,...,A_n\in G/Ad(G)=Hom(\mathbb{Z},G)/G=Hom(\pi_1(S^1),G)/G=\mathcal{R}_{S^1,G}=Z_G(S^1)$ we have*

$$\frac{\#\{\rho\in Hom(\pi_1(\Sigma_{g,n}),G):[\rho|_{C_i}]=A_i\}}{\#G}=Z_G(\Sigma_{g,n})(A_1,...,A_n)$$

$$=G^{2g-2+n}\sum_{\alpha\in irred(G)}\frac{\chi_\alpha(A_1)...\chi_\alpha(A_n)}{Aut(A_1)...Aut(A_n)\dim(\alpha)^{2g-2+n}}$$

Corollary 4.1.14. *(Mednykh's Formula)*
Let $\Sigma_{g,0}$ be the closed surface of genus g, G a finite group and let $irred(G)$ be the finite dimensional irreducible $\mathbb{C}$-representations of G. Consider Z_G the Dijkgraaf-Witten TQFT at dimension $1+1$. We have the following formula

$$\frac{\#Hom(\pi_1(\Sigma_{g,0}),G)}{\#G}=Z_G(\Sigma_{g,0})=G^{2g-2}\sum_{\alpha\in irred(G)}\frac{1}{\dim(\alpha)^{2g-2}}$$

Proof. The LHS of the equality was shown in theorem 4.1.12.

We now want to use the explicit properties of $(1+1)$-dimensional TQFTs. Recall from section G.3 the following properties.

- The information of a $(1+1)$-dimensional TQFT is the same as that of a commutative Frobenius algebra.
- The multiplication is determined by the pair of pants and the cylinder.
- The metric is determined by the cylinder.
- The unit is determined by the disk.

To recalculate the Dijkgraaf-Witten $(1+1)$-dimensional TQFT we must therefore calculate $Z_G(\Sigma_{0,3})$, $Z_G(\Sigma_{0,2})$ and $Z_G(\Sigma_{0,1})$.

Firstly let $A_1, A_2, A_3 \in G/Ad(G) = \mathcal{R}_{S^1,G}$. We want to think of $A_1, A_2, A_3 \in Z(\mathbb{C}G)$ the center on the group algebra in the standard way. That is take $A_1 = \sum_{g \in A_1} g \in Z(\mathbb{C}G)$ etc. Notice that for $\pi_1(\Sigma_{0,n})$ we have

$$\frac{\#\{\rho \in Hom(\pi_1(\Sigma_{0,n}), G) : [\rho|_{C_i}] = A_i\}}{\#G} = \frac{\text{coefficient of } e \text{ in } A_1 ... A_n}{\#G}$$

Therefore we see that

- $Z_G(\Sigma_{0,1})(A_1) = \begin{cases} \frac{1}{\#G} & \text{if } A_1 = e \\ 0 & \text{otherwise} \end{cases}$
- $Z_G(\Sigma_{0,2})(A_1, A_2) = \begin{cases} \frac{\#A_1}{\#G} & \text{if } A_1 = A_2^{-1} \\ 0 & \text{otherwise} \end{cases}$
- $Z_G(\Sigma_{0,3})(A_1, A_2, A_3) = \frac{\text{coefficient of } e \text{ in } A_1 A_2 A_3}{\#G}$

Note that (coefficient of e in $A_1 A_2 A^{-1}$) = (coefficient of A in $A_1 A_2 A^{-1} A$) = (coefficient of A in $A_1 A_2$) $\times \#A$. From these formulas we can see that

$$\sum_{A = A_3^{-1} \in \mathcal{R}_{S^1,G}} Z_G(\Sigma_{0,3})(A_1, A_2, A) \frac{A_3}{Z_G(\Sigma_{0,2})(A, A_3)}$$

$$= \sum_{A \in \mathcal{R}_{S^1,G}} \frac{\text{coefficient of } e \text{ in } A_1 A_2 A^{-1}}{\#A} A$$

$$= \sum_{A \in \mathcal{R}_{S^1,G}} \frac{(\text{coefficient of } A \text{ in } A_1 A_2) \times \#A}{\#A} A = A_1 A_2 \in Z(\mathbb{C}G)$$

Notice that this means not only can we think of $Z_G(S^1) = Z(\mathbb{C}G)$ as vector spaces but in fact $Z_G(S^1) = Z(\mathbb{C}G)$ as algebras. Recall of course that class functions on a finite group correspond to the center of the group algebra. Notice that $Z_G(\Sigma_{0,2})$ is the standard inner product on class functions. Notice that $Z_G(\Sigma_{0,1})$ corresponds to the unit in $Z(\mathbb{C}G)$ as well.

We see that center of the group algebra $Z(\mathbb{C}G)$ is the Frobenius algebra associated to the $(1+1)$-dimensional Dijkgraaf-Witten TQFT Z_G.

From the study of irreducible representations of finite groups we see that $Z(\mathbb{C}G)$ is semi-simple and in fact the idempotent basis is given by $\frac{\#G\chi_\alpha}{\dim(\alpha)}$ for $\alpha \in irred(G)$.

(See [Art91] for an introduction to finite group representations.)

From our calculations in G.3 we have the following formula given the basis of idempotents $\left\{\frac{\#G\chi_\alpha}{\dim(\alpha)}\right\}_{\alpha \in irred(G)}$ where we note that $\left\langle \frac{\#G\chi_\alpha}{\dim(\alpha)}, \frac{\#G\chi_\alpha}{\dim(\alpha)} \right\rangle = \frac{\#G^2}{\dim(\alpha)^2}$.

$$Z_G(\Sigma_{g,n})(A_1, ..., A_n) = \#G^{2g-2+n} \sum_{\alpha \in irred(G)} \frac{\chi_\alpha(A_1)...\chi_\alpha(A_n)}{Aut(A_1)...Aut(A_n)\dim(\alpha)^{2g-2+n}}$$

$\square$

Remark: These relations in group theory don't necessarily say anything too deep about a relation to the topology of 2 manifolds. Considering that the fundamental group of a 2 manifold is free besides one commutator relation, it follows that the representations should be closely related to the structure of the group. The main point is that both the group algebra and $(1+1)$-dimensional cobordisms possess Frobenius structures.

Remark: Notice the similarity between Mednykh's formula and Witten's formula 3.2.1 for the volumes of the moduli space of flat connections. This seems to put Witten's result as a clear generalisation of Mednykh's formula to the case of connected simple Lie groups. This motivates the next section 4.2.

In dimension $(1+1)$ we have calculated the Dijkgraaf-Witten TQFT with respect to some group theoretic quantities. This used the classification of $(1+1)$-dimensional TQFTs. In dimension $(2+1)$ we don't have this luxury so we will build up some examples.

Example: (Simple Calculations in $(2+1)$-dimensions)

- Let S_g be the genus g handle-body. Then $\Sigma_g = \partial S_g$. We have

  $$Z_G(\Sigma_g) = Span_{\mathbb{C}}\{(a_1, b_1, ..., a_g, b_g) \in G^{2g} : a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1} = e\}/G$$

  Then for $[a_1, b_1, ..., a_g, b_g] \in G^{2g}/G$ such that $a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1} = e$.

  $$Z_G(S_g, \iota, \varnothing) : Z_G(\Sigma_g) \to \mathbb{C} \quad s.t$$

  $$Z_G(S_g, \iota, \varnothing)[a_1, b_1, ..., a_g, b_g] = \begin{cases} \frac{1}{\#Stab_G(a_1, b_1, ..., a_g, b_g)} & \text{if for all } i \text{ we have } b_i = e \\ 0 & \text{otherwise} \end{cases}$$

- Notice that $(S_g \sqcup S_g, \iota \sqcup \iota, \varnothing) \circ (\Sigma_g \times [0,1], \varnothing, \iota \sqcup \iota) = \big((S^2 \times S^1)^{g\#}, \varnothing, \varnothing\big)$. Let $(a_1, b_1, ..., a_g, b_g) = (a, b)$, $(a, e) = (a_1, e, ..., a_g, e)$ and

$$R = \{(a_1, b_1, ..., a_g, b_g) \in G^{2g} : a_1 b_1 a_1^{-1} b_1^{-1} ... a_g b_g a_g^{-1} b_g^{-1} = e\}/G$$

We then have

$$Z_G((S^2 \times S^1)^{g\#}) = \sum_{(a,b)\in R} Z_G(S_g, \iota, \varnothing)(a,b) Z_G(S_g, \iota, \varnothing)(a,b) \# Stab_G(a,b)$$

$$= \sum_{a \in G^g/G} \frac{1}{\# Stab_G(a,e)} = \sum_{a \in G^g/G} \frac{Orb_G(a,e)}{\# G} = \frac{\# G^g}{\# G} = \# G^{g-1}$$

This can be calculated directly using theorem 4.1.12 and noting that $\pi_1((S^2 \times S^1)^{g\#}) = F_g$ and therefore $\# Hom(F_g, G) = \# G^g$ and so $Z_G((S^2 \times S^1)^{g\#}) = \frac{\# G^g}{\# G} = \# G^{g-1}$.

- Suppose $f : \Sigma_g \to \Sigma_g$ swap the a and b cycles on the surface. For example for $g = 1$ the isomorphism that switches factors $S^1 \times S^1 \cong S^1 \times S^1$. Then notice $(S_g \sqcup S_g, \iota \sqcup f, \varnothing) \circ (\Sigma_g \times [0,1], \varnothing, \iota \sqcup \iota) = (S^3, \varnothing, \varnothing)$. Now

$$Z_G(S_g, f, \varnothing)[a_1, b_1, ..., a_g, b_g] = \begin{cases} \frac{1}{\# Stab_G(a_1, b_1, ..., a_g, b_g)} & \text{if for all } i \text{ we have } a_i = e \\ 0 & \text{otherwise} \end{cases}$$

This means

$$Z_G(S^3) = \sum_{(a,b)\in R} Z_G(S_g, \iota, \varnothing)(a,b) Z_G(S_g, f, \varnothing)(a,b) \# Stab_G(a,b)$$

$$= \frac{Stab(e,e)}{Stab(e,e)^2} = \frac{1}{Stab(e,e)} = \frac{1}{\# G}$$

Notice that using theorem 4.1.12 we have $Z_G(S^3) = \frac{\# Hom(\pi_1(S^3), G)}{\# G} = \frac{1}{\# G}$.

Remark: Throughout this section we have explicitly used the fact that G is finite. To understand what happens when we replace our finite group with a Lie group G in general requires work. For those interested consult [FHLT09]. In the next section we will exploit the classification of $(1+1)$-dimensional TQFTs to construct an analogue of the Dijkgraaf-Witten TQFT in dimension $(1+1)$ with Lie groups.

Remark: The definition 4.1.2 means that G-covers can be glued together. This allows one to define the subcategory of $\underline{Cob_{d+1}}$ given by the G-covers. This will be denoted $\underline{GCov_{d+1}}$. There is a functor $Base : \underline{GCov_{d+1}} \to \underline{Cob_{d+1}}$ which takes a G-cover to its base. There is a map $Aut : Hom_{\underline{GCov_{d+1}}}(P_1, P_2) \to \frac{1}{\# G}\mathbb{Z}_{>0}$ such that for $P \in Hom_{\underline{GCov_{d+1}}}(P_1, P_2)$ we have $Aut(P) = \frac{\# Aut(P_2)}{\# Aut(P)}$. Notice that $\# Aut(P)$ is related to the pre-image $Base^{-1}(\Sigma \times [0,1])$.

For $\Sigma \in ob(\underline{Cob_{d+1}})$ we define $Z_G(\Sigma) = Span_{\mathbb{C}}\{Base^{-1}(\Sigma)\}$ and

$$Z_G(M)(P_1) = \sum_{\substack{P \in Base^{-1}(M) \\ s.t \quad P|_{\Sigma_1} = P_1}} Aut(P) P|_{\Sigma_2}$$

To generalise Dijgkraaf-Witten TQFT to Lie groups these kinds of categories and functors could be important.

4.2 Recursions and Topological Quantum Field Theory

Covering spaces give rise to a completely rigorous TQFT with a partition function being defined as a finite sum of G-covers on a given cobordism.

Although this is rigorous, the case of most interested is when we replace finite group with a Lie group. This will be the case of interest when we study gauge theory with a Lie group symmetry on a manifold.

Notice that representation classes of the fundamental group into a Lie group are described by the moduli space of flat connections. Therefore, we wish to define a measure on the moduli space in order to define the partition function and to calculate the associated invariants.

Atiyah and Bott have defined such a measure and this is exactly the content of section 2.1.2. This defines some measure on the moduli space of flat connections and in fact Witten has calculated the volume associated to this measure as shown in theorem 3.2.1.

We want to understand how these volumes could correspond to a partition function of some TQFT.

4.2.1 Volume Recursions with a Canonical Basis of Class Functions

A trivial topological gauge theory should take d dimensional manifolds the free vector space of flat connections on the manifolds. It should take $(d+1)$-dimensional manifolds as functions that take in boundary conditions and give numbers or in particular the volume of the moduli space of flat connections. This is exactly the definition of the Dijkgraaf-Witten TQFT.

When $d = 1$ the vector space associated to the circle as spanned by flat connections on the circle which is the set of conjugacy classes. This is closely related to the class functions from the Lie group. The issue on the outset is that the dimension of $Z(S^1)$ is infinite and this will lead to $Z(S^1 \times S^1) = \dim(Z(S^1)) = \infty \in \mathbb{C}$.

(1 + 1)-dimensional TQFTs correspond to Frobenius algebra's over some module. In this section we will study the $L^2(G)$ functions on a compact group and show that they illustrate various properties of a Frobenius algebra over $\mathbb{C}$. This will be analogous to the group algebra $\mathbb{C}[G]$ of a finite group.

Definition: ($L^2(G)$ Banach Algebra)
Let G be a compact Lie group. Using the Haar measure on G we can define the Banach space $L^2(G)$. $L^2(G)$ has an inner product such that for $f, g \in L^2(G)$ we have

$$\langle f, g \rangle = \int_G f(x^{-1})g(x)dx$$

We can also define a product structure on $L^2(G)$ via convolution. For $f, g \in L^2(G)$ we have

$$f \star g(x) = \int_G f(h)g(h^{-1}x)dh$$

This space is not quite the space we are interested in and we will consider the subspace of class functions and see if the inner product $\langle -, - \rangle$ and $\star$ satisfy the Frobenius condition.

Definition: (Z_G)
Let G be a compact semi-simple Lie group. Let $Z_G \subseteq L^2(G)$ be the subset of class functions (i.e $f(yxy^{-1}) = f(x)$). This is a sub-algebra of $L^2(G)$. Notice that $Z_G \subseteq \{f : \mathcal{R}_{S^1,G} \to \mathbb{C}\}$ is the subset of square integrable functions.

Theorem 4.2.1. *(Canonical Basis)*
Let G be a compact semi-simple Lie group. Let $irred(G)$ be the set of finite dimensional irreducible representations and for $\alpha \in irred(G)$ let the character of the representation be denoted $\chi_\alpha : G \to \mathbb{C}$.

Then $\{\chi_\alpha\}_{\alpha \in irred(G)}$ is an orthonormal Schauder basis of Z_G with respect to the inner product $\langle -, - \rangle$ on $L^2(G)$. That is for $\alpha, \beta \in irred(G)$ we have

$$\langle \chi_\alpha, \chi_\beta \rangle = \int_G \chi_\alpha(x^{-1})\chi_\beta(x)dx = \delta_{\alpha,\beta}$$

Moreover with the convolution product $\star$ we have

$$\chi_\alpha \star \chi_\beta = \delta_{\alpha,\beta}\frac{Vol(G)}{\dim(\alpha)}\chi_\alpha$$

Proof. See the theorem of Peter-Weyl theorem B.2.2 and section II 4 of [Bt03]. □

Corollary 4.2.2. *(Z_G is a Non-Unital Frobenius Banach Algebra)*
For $f, g, h \in Z_G$ we have

$$\langle f \star g, h \rangle = \langle f, g \star h \rangle$$

Proof. By the previous theorem we know that $\{\chi_\alpha\}_{\alpha \in irred(G)}$ is a basis so we need only check this result on basis elements. The theorem also gives the following

$$\langle \chi_\alpha \star \chi_\beta, \chi_\gamma \rangle = \delta_{\alpha,\beta,\gamma} \frac{Vol(G)}{\dim(\alpha)} = \langle \chi_\alpha, \chi_\beta \star \chi_\gamma \rangle$$

□

Definition: (Volumes as Linear Maps)
Let G be a compact semi-simple Lie group let F be Weyl denominator of theorem B.2.8. Define the following linear maps

$$Z_{G,g,n} : Z_G^{\otimes n} \to \mathbb{C} \quad s.t \quad Z_{G,g,n}(f_1, ..., f_n) = Vol(G)^{-n}$$

$$\int_{G/Ad(G)} ... \int_{G/Ad(G)} V_{G,g,n}(C_1^{-1}, ..., C_n^{-1}) f_1(C_1)...f_n(C_n)\sqrt{F(C_1)...F(C_n)} dC_1...dC_n$$

Remark: The volumes as calculated by Witten are not well defined for all Euler characteristics as his expressions have divergences. Therefore the above linear map is not always defined however we will abstract and avoid the issues associated with convergence. This is the same reason that Z_G is non-unital.

Remark: We would like to define a $(1+1)$-dimensional TQFT Z_G with $Z_G = Z_G(S^1)$ and $Z_{G,g,n} = Z(\Sigma_{g,n})$. However we want to be able to extract the volume from these linear maps $Z(\Sigma_{g,n})$. This won't be possible without Dirac δ-functions which are not in $L^2(G)$.

Lemma 4.2.3. *(Convolution Product vs. $Z_{G,0,3}$)*
Let G be a compact semi-simple Lie group. Then for $f, g, h \in Z_G(S^1)$ we have

$$\langle f \star g, h \rangle = \frac{\#Z(G)}{Vol(T)^3} Z_G(\Sigma_{0,3})(f, g, h)$$

Proof. We will prove this fact on the basis of characters χ_α and this will be enough. Now we know

$$\langle \chi_\alpha \star \chi_\beta, \chi_\gamma \rangle = \delta_{\alpha,\beta,\gamma} \frac{Vol(G)}{\dim(\alpha)}$$

From theorem 3.2.6 we know that

$$V_{G,0,3}(C_1, C_2, C_3)$$

$$= \frac{\#Z(G)Vol(G)}{Vol(T)^3} \sum_{\alpha \in irred(G)} \frac{\chi_\alpha(C_1)\chi_\alpha(C_2)\chi_\alpha(C_3)\sqrt{F(C_1)F(C_2)F(C_3)}}{\dim(\alpha)}$$

This means that using the Weyl integration formula theorem B.2.6 we have

$$Z_G(\Sigma_{0,3})(\chi_\alpha, \chi_\beta, \chi_\gamma) = Vol(G)^{-3} \frac{\#Z(G)Vol(G)}{Vol(T)^3} \sum_{\epsilon \in irred(G)} \frac{Vol(G)^3 \delta_{\alpha,\epsilon}\delta_{\beta,\epsilon}\delta_{\gamma,\epsilon}}{\dim(\epsilon)}$$

$$= \delta_{\alpha,\beta,\gamma} \frac{\#Z(G)Vol(G)}{Vol(T)^3 \dim(\alpha)}$$

□

Remark: (Conjugacy Classes Almost a Canonical Basis)
To extract the volumes from $Z_{G,g,n}$ we need to allow δ-functions centred at a given conjugacy class in Z_G. This brings some complications so we will instead take a formal version of the $L^2(G)$ algebra which will we then use to construct a $(1+1)$-dimensional TQFT.

4.2.2 Graded Topological Quantum Field Theories and 1+1-Dimensional Gauge Theories

In this section a formal version or analogue of $L^2(G)$ will be defined and will be shown to capture the information of the volume of the moduli space of flat connections.

Definition: (Formal Space of Characters)
Let G be a compact semi-simple Lie group. Let $irred(G)$ be the set of finite dimensional irreducible representations. Let

$$Z_G(S^1) = \mathbb{C}^{irred(G)}$$

be the one dimensional $\mathbb{C}^{irred(G)}$-module given by component wise multiplication.

Remark: This ring should be viewed as formal sums of the irreducible characters χ_α. Note however that the product is not quite the convolution.

Definition: (Representation of the Conjugacy Classes)
Let G be a compact semi-simple Lie group. Let $C \in G/Ad(G)$. Then we can take

$$C \in Z_G(S^1)$$

such that for $\alpha \in irred(G)$ we have $C_\alpha = \chi_\alpha(C)$.

Definition: (Linear Maps related to Volumes)
Let G be a compact semi-simple Lie group. Then define

$$Z_G(\Sigma_{g,n}) : Z_G(S^1)^{\otimes n} \to \mathbb{C}^{irred(G)} \quad s.t$$

$$Z_G(\Sigma_{g,n})(v_1, ..., v_n)_\alpha = \frac{Vol(G)^{2g-2+n}}{\dim(\alpha)^{2g-2+n}}(v_1)_\alpha ...(v_n)_\alpha$$

Theorem 4.2.4. *(Topological Gauge Theory as a TQFT)*
Let G be a compact semi-simple Lie group. Then Z_G defines a $(1+1)$-dimensional TQFT over the ring $\mathbb{C}^{irred(G)}$.

Proof. To prove that Z_G is a TQFT we must check that it satisfies the Frobenius object conditions. That is we must check $Z_G(\Sigma_{0,2})$ and defines a product, $Z_G(\Sigma_{0,3})$ defines a product with unit defined $Z_G(\Sigma_{0,1})$.

For $(u_\alpha)_{\alpha \in irred(G)}, (v_\alpha)_{\alpha \in irred(G)} \in Z_G(S^1)$ we have

$$\langle (u_\alpha)_{\alpha \in irred(G)}, (v_\alpha)_{\alpha \in irred(G)} \rangle = Z_G(\Sigma_{0,2})((u_\alpha)_{\alpha \in irred(G)}, (v_\alpha)_{\alpha \in irred(G)})_\alpha$$

$$= \frac{Vol(G)^{2\times 0-2+2}}{\dim(\alpha)^{2\times 0-2+2}} u_\alpha v_\alpha = u_\alpha v_\alpha$$

This is a non-degenerate inner product. We have the following basis for $Z_G(S^1)$ given by $\mathbb{I}$ such that $\mathbb{I}_\alpha = 1$ for all $\alpha \in irred(G)$. Notice that $Z_G(\Sigma_{0,2})(\mathbb{I}, \mathbb{I}) = \mathbb{I}$.

We now want to check that $Z_G(\Sigma_{0,3})$ defines a product. This needs to contract using $Z_G(\Sigma_{0,2})$. Notice that as $Z_G(S^1)$ is 1-dimensional that the sum in the contraction has only one term. We can define a product as follows

$$(u_\alpha)_{\alpha\in irred(G)} \cdot (v_\alpha)_{\alpha\in irred(G)})_\alpha = \left(Z_G(\Sigma_{0,3})((u_\alpha)_{\alpha\in irred(G)}, (v_\alpha)_{\alpha\in irred(G)})_\alpha, \mathbb{I}) Z_G(\mathbb{I}, \mathbb{I})^{-1}\mathbb{I}\right)_\alpha$$
$$= \frac{Vol(G)^{2\times 0-2+3}}{\dim(\alpha)^{2\times 0-2+3}} u_\alpha v_\alpha = \frac{Vol(G)}{\dim(\alpha)} u_\alpha v_\alpha$$

Notice that $(u_\alpha)_{\alpha\in irred(G)} \cdot (v_\alpha)_{\alpha\in irred(G)})_\alpha$ is associative and commutative. Also notice that as Z_G is one dimensional this product is completely determined by $(\mathbb{I} \cdot \mathbb{I})_\alpha = \frac{Vol(G)}{\dim(\alpha)}$. We now need to check the Frobenius condition. We have

$$\langle (u_\alpha)_{\alpha\in irred(G)} \cdot (v_\alpha)_{\alpha\in irred(G)}, (w_\alpha)_{\alpha\in irred(G)} \rangle_\alpha = \frac{Vol(G)}{\dim(\alpha)} u_\alpha v_\alpha w_\alpha$$
$$= \langle (u_\alpha)_{\alpha\in irred(G)}, (v_\alpha)_{\alpha\in irred(G)} \cdot (w_\alpha)_{\alpha\in irred(G)} \rangle_\alpha$$

The last check is that $Z_G(\Sigma_{0,1})$ induces a unit as follows.

$$\left(Z_G(\Sigma_{0,1})(\mathbb{I}) Z_{0,2}(\mathbb{I}, \mathbb{I})^{-1}\mathbb{I}\right)_\alpha = \frac{Vol(G)^{2\times 0-2+1}}{\dim(\alpha)^{2\times 0-2+1}} = \frac{\dim(\alpha)}{Vol(G)}$$

Now notice that

$$\left(Z_G(\Sigma_{0,1})(\mathbb{I}) Z_{0,2}(\mathbb{I}, \mathbb{I})^{-1}\mathbb{I} \cdot (u_\alpha)_{\alpha\in irred(G)}\right)_\alpha = \frac{Vol(G)}{\dim(\alpha)} \frac{\dim(\alpha)}{Vol(G)} u_\alpha = u_\alpha$$

We have checked all the conditions and therefore Z_G does indeed define a TQFT. ☐

Definition: $\left(\ell^1\left(\mathbb{C}^{irred(G)}\right)\right)$

$$\ell^1\left(\mathbb{C}^{irred(G)}\right) = \left\{ (v_\alpha)_{\alpha\in irred(G)} \in \mathbb{C}^{irred(G)} : \sum_{\alpha\in irred(G)} |v_\alpha| < \infty \right\}$$

The we can define a map $\mu : \ell^1\left(\mathbb{C}^{irred(G)}\right) \to \mathbb{C}$ such that for $(v_\alpha)_{\alpha\in irred(G)} \in \ell^1\left(\mathbb{C}^{irred(G)}\right)$

$$\mu\left((v_\alpha)_{\alpha\in irred(G)}\right) = \sum_{\alpha\in irred(G)} v_\alpha$$

Theorem 4.2.5. *(Relation to Volumes of the Moduli Space of Flat Connections) Let G be a compact semi-simple Lie group and F be the Weyl denominator of theorem B.2.8. For $C_1, ..., C_n \in Z_G(S^1)$ representing the conjugacy classes $C_1, ..., C_n \in G/Ad(G) \cong \mathcal{R}_{S^1,G}$. If $V_{G,g,n}(C_1, ..., C_n)$ is finite and $Z_G(\Sigma_{g,n})(C_1, ..., C_n) \in \ell^1\left(\mathbb{C}^{irred(G)}\right)$ then*

$$V_{G,g,n}(C_1, ..., C_n) = \frac{\#Z(G)\sqrt{F(C_1)...F(C_n)}}{Vol(T)^n} \mu\left(Z_G(\Sigma_{g,n})(C_1, ..., C_n)\right)$$

Proof. This follows directly from the definition of $\mu\left(Z_G(\Sigma_{g,n})(C_1, ..., C_n)\right)$ and theorem 3.2.1. ☐

Appendices

Appendix A

Smooth Manifolds, Fibre Bundles and Tangent Bundles

Will define these basic objects in the theory of differential topology. For more details and examples consult either [Spi70] or [KN63]. For vector bundles specifically one may consult [MS74] for the basic definitions and results. All the following definitions try to make rigorous the ideas of local differentiation and integration in a topological space that looks locally like $\mathbb{R}^n$.

A.1 Smooth Manifolds

Definition: (Manifold)
Let M be a Hausdorff and second countable topological space. If for each $x \in M$ there exists

- an open set $U \subseteq M$ with $x \in U$
- a open set $\Omega \subseteq \mathbb{R}^n$
- and a homeomorphism $\varphi : \Omega \to U$

we call M an n-**manifold**.

Remark: This definition says that M is locally homeomorphic to $\mathbb{R}^n$. Note also that the dimension n is an invariant of the manifold.

Notation: Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $C^r(\Omega, \mathbb{R}^m)$ be the set of continuous functions such that all partial derivative up to order r are continuous.

$C^0(\Omega, \mathbb{R}^m)$ will be continuous functions and $C^\infty(\Omega, \mathbb{R}^m)$ will be the smooth functions (i.e all partial derivative exists and are continuous).

Definition: (C^r Atlas)
Let M be a n-manifold. Let I is some indexing set and for $i \in I$ let

- an open set $U_i \subseteq M$

- a open set $\Omega_i \subseteq \mathbb{R}^n$
- and a homeomorphism $\varphi_i : \Omega_i \to U_i$

such that $\bigcup_{i \in I} U_i = M$. If for $i, j \in I$ we have $\varphi_j^{-1} \circ \varphi_i|_{\varphi_i^{-1}(U_i \cap U_j)}$ a C^r map between subsets of $\mathbb{R}^n$ then we say $\{(U_i, \varphi_i)\}_{i \in I}$ is a C^r **atlas** for M and (U_i, φ_i) a **chart** in the atlas.

$$\begin{tikzcd} & M & \\ \mathbb{R}^n \supseteq \Omega_i \arrow[ur, hook, "\varphi_i"] \arrow[rr, dashed] & & \Omega_j \subseteq \mathbb{R}^n \arrow[ul, hook', "\varphi_j"'] \end{tikzcd}$$

Definition: (C^r Manifold)
Let M be a n-manifold. A C^r **structure** on M is an equivalence class of C^r atlases. Two atlases $\{(open_i, \varphi_i)\}_{i \in I}$ and $\{(V_j, \phi_j)\}_{j \in J}$ are equivalent if $\{(U_i, \varphi_i)\}_{i \in I} \cup \{(V_j, \phi_j)\}_{j \in J}$ is a C^r atlas. An n-manifold and its C^r structure is called a C^r n**-manifold**.

An n-manifold with a C^0 structure will be called a topological n-manifold and a n-manifold with a C^∞ structure will be called a smooth n-manifold.

Definition: (C^r Maps)
Let M be C^r n-manifold and N be C^r m-manifold with atlases in their C^r structures $\{(U_i, \varphi_i)\}_{i \in I}$ and $\{(V_j, \phi_j)\}_{j \in J}$ respectively. If $\phi_j^{-1} \circ f \circ \varphi_i|_{\varphi_i^{-1}(f^{-1}(V_j))}$ is a C^r between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ for all $i \in I$ and $j \in J$ then we say f is a C^r **map** from M to N.

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ \mathbb{R}^n \supseteq U_i \arrow[u, hook, "\varphi_i"] \arrow[r, dashed] & V_j \subseteq \mathbb{R}^m \arrow[u, hook, "\phi_j"] \end{tikzcd}$$

Let $C^r(M, N)$ be the set of C^r maps from M to N and $C^r(M)$ be the real commutative algebra of smooth maps from M to $\mathbb{R}$ with point wise addition and multiplication.

Definition: (Diffeomorphism)
If M and N are smooth manifolds and $f : M \to N$ is a smooth bijection with smooth inverse then we say that f is a **diffeomorphism**.

Remark: It can be seen that C^r manifolds and C^r maps form a category. In particular we are interested in the category of smooth manifolds and smooth maps.

Definition: (Manifolds with Boundary)
Let M be a Hausdorff and second countable topological space. If for each $x \in M$ there exists

- an open set $U \subseteq M$ with $x \in U$
- a open set $\Omega \subseteq \mathbb{R}^{n-1} \times \mathbb{R}_{\geqslant 0}$
- and a homeomorphism $\varphi : \Omega \to U$

we call M an **n-manifold with boundary**.

Remark: For a manifold with boundary M we can define the boundary of M denoted ∂M. Then $M - \partial M$ is a manifold in the ordinary sense and we can define C^r structures on manifolds with boundary.

A.2 Smooth Bundles and Tangent Bundles

We wish to define that tangent space of a smooth manifold at a point and then define the tangent bundle and induced maps between tangent bundles. Firstly, we will consider the definition of a fibre bundle. For more information on fibre bundles consult the classic text [Ste51].

Definition: (Fibre Bundle)
Let E, B and F be topological spaces and $\pi : E \to B$ a surjective map. If for every point $x \in B$ there exists an open set $U \subseteq B$ with $x \in U$ such that $\varphi : \pi^{-1}(U) \cong U \times F$ with $proj_U \circ \varphi = \pi$ we say that $(\pi : E \to B, F)$ is a **fibre bundle**.

If E, B, F are smooth manifolds with π a smooth map we say that $(\pi : E \to B, F)$ is a **smooth fibre bundle**.

Definition: (Bundle Map)
Let $(\pi_1 : E_1 \to B_1, F_1)$ and $(\pi_2 : E_2 \to B_2, F_2)$ be fibre bundles. Let $f : B_1 \to B_2$ be a continuous map. Then a continuous map $g : E_1 \to E_2$ is called a **bundle map** covering f if $\pi_2 \circ g = f \circ \pi_1$.

Definition: (Sections)
Let $(\pi : E \to B, F)$ be a fibre bundle. A **section** is a continuous map $s : B \to E$ such that $\pi \circ s = id$. For smooth fibre bundles we take smooth maps s.

Definition: (Real Vector Bundle)
Let V be a topological vector space over $\mathbb{R}$. Suppose we have a fibre bundle $(\pi : E \to B, V)$ such that for $x \in B$ we have the structure of vector space $\pi^{-1}(x) \cong V$. If for each $x \in B$ there exists an open set $U \subseteq B$ with $x \in U$ with a homeomorphism $\varphi : U \times V \cong \pi^{-1}(U)$ such that $proj_U = \pi \circ \varphi$ such that $\varphi(x, -) : V \to \pi^{-1}(x)$ is an isomorphism of vector spaces then we call $(\pi : E \to B, V)$ a **real vector bundle**.

Remark: Smooth vector bundles can be defined by requiring all spaces and maps to be smooth. We can also analogously define complex vector bundles by choosing V a complex vector space. The underlying theme of any bundle is local triviality. In other words, locally we have some Cartesian product of spaces potentially with some additional structure.

Notation: If $(\pi : E \to B, V)$ is a vector bundle. For $x \in B$ we denote the fibre above x with its vector space structure as E_x. We will right elements of E_x with a subscript as follows as $v_x \in E_x$.

There is a natural vector space associated to a vector bundle $(\pi : E \to B, V)$ defined as follows.

Definition: (Space of Sections)
Let $(\pi : E \to B, V)$ be a vector bundle. Consider the set of sections. We then define addition and scalar multiplication as follows for sections $s_1, s_2 : B \to E$ and $r \in \mathbb{R}$.

- $s_1 + s_2 : B \to E$ such that $(s_1 + s_2)(x) = s_1(x) + s_2(x)$
- $rs_1 : B \to E$ such that $(rs_1)(x) = r(s_1(x))$

The local triviality condition means that these maps will indeed be continuous and in fact smooth for smooth fibre bundles. We define $\Gamma(E)$ to be the **space of sections** of E.

There are two natural vector bundles associated to every smooth manifolds.

Definition: (Derivations)
Let M be a smooth manifold and $x \in M$. Consider the following vector space where $C^\infty(M)^*$ denote the dual of the vector space $C^\infty(M)$.

$$T_xM = \{D_x : C^\infty(M)^* : D_x(fg) = D_x(f)g(x) + f(x)D_x(g)\}$$

This is called the **space of derivations** on M at the point $x \in M$.

Remark: Amazingly, the dimension of T_xM is the same as the dimension of M. There is a geometric interpretation of these derivations as directional derivative and this definition canonically coincides with more geometric definitions of tangent vectors so we will refer to T_xM as the tangent space of M at $x \in M$.

Notation: Let $T_x^*M = (T_xM)^*$. We call this the cotangent space of M at $x \in M$.

Lemma A.2.1. *Let M be a smooth n-manifold, $x \in M$, $U \subseteq M$ and $\Omega \subseteq \mathbb{R}^n$ be open with $x \in U$, and $\varphi : \Omega \to U$ a diffeomorphism. Then the derivations $\frac{\partial}{\partial \varphi_i}|_x$ defined by $\frac{\partial}{\partial \varphi_i}|_x : C^\infty M \to \mathbb{R}$ such that $\frac{\partial}{\partial \varphi_i}|_x(f) = \frac{\partial}{\partial x_i}(f \circ \varphi)|_{\varphi^{-1}(x)}$ form a basis for T_xM.*

Notation: Let $d\varphi_i|_x$ be the dual basis element to $\frac{\partial}{\partial \varphi_i}|_x$.

Definition: (Induced Maps)
Let M and N be smooth manifolds. Let $f : M \to N$ be a smooth map. For $x \in M$ consider T_xM, $T_{f(x)}N$, T_x^*M and $T_{f(x)}^*N$. We can define the following maps from T_xM to $T_{f(x)}N$ and $T_{f(x)}^*N$ to T_x^*M induced by the map f as follows for $g \in C^\infty(M)$ and $h \in C^\infty(N)$ noting that $h \circ f \in C^\infty(M)$.

$$T_xf = df_x = (f_*)_x : T_xM \to T_{f(x)}N \quad s.t \quad ((f_*)_x(v_x))(h) = v_x(h \circ f)$$

$$T_x^*f = (f^*)_x : T_{f(x)}^*N \to T_x^*M \quad s.t \quad (f^*)_x(\omega_x) = \omega_x \circ (f_*)_x$$

Remark: Rigorously defining all the categories we can make T_x a covariant functor from based smooth manifolds to real vector spaces and T_x^* a contravariant functor from based smooth manifolds to real vector spaces.

Definition: (Tangent Bundle)
Let M be a smooth n-manifold. Let $TM = \bigsqcup_{x \in M} T_xM$ as a set. Define a topology and smooth structure on TM as follows. For $v \in TM$ we have $v \in T_xM$ for some x and so we have an open set $U \subseteq M$ with $x \in U$, an open set $\Omega \subseteq \mathbb{R}^n$ and a diffeomorphism $\varphi : \Omega \to U$.

Define the following chart $\Phi : \Omega \times \mathbb{R}^n \to \bigsqcup_{u \in U} T_uM$ such that for $\Phi(y, a_1, ..., a_n) = a_1 \frac{\partial}{\partial \varphi_1}|_{\varphi(y)} + ... + a_n \frac{\partial}{\partial \varphi_n}|_{\varphi(y)}$.

So TM inherits the structure of a smooth manifold. The canonical projection $\pi : TM \to M$ such that for $v \in T_xM$ we have $\pi(v) = x$ is smooth and makes TM into a vector bundle we call the **tangent bundle** of M.

Define the following chart $\Psi : \Omega \times \mathbb{R}^n \to \bigsqcup_{u \in U} T_u^*M$ such that for $\Phi(y, a_1, ..., a_n) = a_1 d\varphi_i|_y + ... + a_n d\varphi_i|_y$.

So T^*M inherits the structure of a smooth manifold. The canonical projection $\pi^* : T^*M \to M$ such that for $v \in T_x^*M$ we have $\pi^*(v) = x$ is smooth and makes T^*M into a vector bundle we call the **cotangent bundle** of M.

Definition: (Induced Maps)
Let M and N be smooth manifolds. Let $f : M \to N$ be a smooth map. M, TN, T^*M and T^*N. We can define the following maps from TM to TN and T^*N to T^*M induced by the map f as follows.

$$Tf = df = f_* : TM \to TN \quad s.t \quad f_*(v_x) = (f_*)_x(v_x)$$

$$T^*f = f^* : T^*N \to T^*M \quad s.t \quad f^*(\omega) = \omega \circ f_*$$

Remark: Rigorously defining all the categories we can make T a covariant functor from smooth manifolds to smooth real vector bundles and T^* a contravariant functor from smooth manifolds to smooth real vector bundles.

A.3 Vector Fields, Differential Forms and Integration

The cotangent bundle is important as we can analogously define the k-th exterior power of the cotangent bundle $\Lambda^k T^*M$. We then use this to define differential forms which we use to define integration on a manifold. For the functorial constructions of vector bundles such as $\Lambda^k T^*M$ a complete description is given in section 3f of [MS74]. For a great introduction to de Rham cohomology and differential forms see [BT82].

Definition: (Differential Forms)
The space of **k-forms** is defined as $\Omega^k(M) = \Gamma(\Lambda^k T^*M)$.

Definition: (Vector Fields and Flow)
Let M be a smooth manifold. Let $\mathfrak{X}(M) = \Gamma(TM)$ be the space of **vector fields**. Let $X \in \mathfrak{X}(M)$. From existence and uniqueness results in the study of differential equations we see that given a point $x \in M$ there exists a unique path connected subset $0 \in \Omega \subseteq \mathbb{R}$ and a path $\gamma : \Omega \to M$ such that $\gamma(0) = x$ and $T_r\gamma(v_r) = X_{\gamma(r)}$.

Remark: We can view $\Omega^k(M)$ as antisymmetric k-linear functions from the space of vector fields $\mathfrak{X}(M)$ to $C^\infty(M)$.

Remark: The induced maps between tangent, cotangent bundles and exterior powers of cotangent bundles induce maps between the sections of these bundles.

Lemma A.3.1. *Let M be a smooth map and a chart $(U, \varphi : \Omega \to U)$. Note that then for $\omega \in \Omega^k(M)$ and $v \in \mathfrak{X}(M)$ we have the following local forms of ω and v for some $f_{i_1,\dots,i_k}, g_i \in C^\infty(\Omega)$ for $i_j, i \in \{1, \dots, n\}$*

$$\varphi^*(\omega|_U) = \sum_{0<i_1<\dots<i_k\leqslant n} f_{i_1,\dots,i_k}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

$$(\varphi^{-1})_* v = \sum_{i=1}^{n} g_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$$

Remark: We will often consider these kinds of objects locally. This is often the way one builds up a k form or an vector field. Taking an atlas for each open set locally they have the concrete form above and then one checks that they agree on intersections. The next definition illustrates this idea.

Definition: (Lie Derivative)
Let M be a smooth manifold and consider $X, Y \in \mathfrak{X}(M)$, $\omega \in \Omega^k(M)$ and let $\varphi : M \times \mathbb{R} \to M$ be the flow generated by X. Define the **lie derivative** of Y and ω in the direction of X as follows.

$$\mathcal{L}_X(Y) = \lim_{t\to 0} \frac{T\varphi_t(Y) - Y}{t}$$

$$\mathcal{L}_X(\omega) = \lim_{t \to 0} \frac{\varphi_t^*(\omega) - \omega}{t}$$

Note that the Lie derivative defines a Lie bracket on the vector fields given by

$$[X, Y] = \mathcal{L}_X(Y)$$

Theorem A.3.2. *(Frobenius' Theorem)*
Let M be a smooth manifold. If $H \subseteq TM$ is a sub-bundle then H is defined by a foliation if and only if H is integrable. That is, H is the tangent bundle of the leaves of a foliation if and only if H is closed under the Lie bracket operation on vector fields on H.

Definition: (Exterior Derivative)
Let M be a smooth manifold and $\omega \in \Omega^k(M)$. The exterior derivative is the unique set of maps such that for $k \in \mathbb{Z}_{\geqslant 0}$ we have

- We have $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ a linear map
- For $f \in \Omega^0(M) = C^\infty(M)$ we have and $d_0 f = df = Tf = f_*$
- $d_{k+1} \circ d_k = 0$
- For $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^p(M)$ we have

$$d_{k+p}(\alpha \wedge \beta) = d_k(\alpha) \wedge \beta + (-1)^k \alpha \wedge d_p(\beta)$$

Equivalently if $\omega \in \Omega^k(M)$ is given locally for a chart (U, φ) by

$$\varphi^*(\omega|_U) = \sum_{0 < i_1 < \ldots < i_k \leqslant n} f_{i_1,\ldots,i_k}(x_1, \ldots, x_n) dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

then $d\omega \in \Omega^{k+1}(M)$ is given by

$$\varphi^*(d\omega|_U) = \sum_{i_{k+1}=1}^{n} \left(\sum_{0 < i_1 < \ldots < i_k \leqslant n} \frac{\partial f_{i_1,\ldots,i_k}}{\partial x_{i_{k+1}}}(x_1, \ldots, x_n) dx_{i_{k+1}} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \right)$$

For vector fields $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$ we have the following where

$$(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{k+1})$$

denotes emission of the element X_i.

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}))$$

$$+ \sum_{1 \leqslant i < j \leqslant k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1})$$

Remark: We call forms $\omega \in \Omega^k(M)$ closed if $d\omega = 0$ and exact if there exists $\beta \in \Omega^{k-1}(M)$ such that $d\beta = \omega$. Notice that all exact forms are closed as $d^2 = d_k \circ d_{k-1} = 0$.

Definition: (De Rham Cohomology)
Let M be a smooth manifold. The i-th de Rham cohomology group is defined as

$$H^i_{dR}(M) = ker(d_i)/im(d_{i-1})$$

Lemma A.3.3. *(Poincaré's Lemma)*
Let M be a smooth manifold. If M is simply connected then for $i > 0$ we have

$$H^i_{dR}(M) = 0$$

In particular if $M \cong B^k = \{x \in \mathbb{R}^k : \|x\| < 1\}$ then for $i > 0$

$$H^i_{dR}(M) = 0$$

We have an association between the Lie derivative and the exterior derivative.

Theorem A.3.4. *(Cartan's Magic Formula)*
Let M be a smooth manifold. For $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$ let $\iota_X : \Omega^*(M) \to \Omega^{*-1}(M)$ such that for $\beta \in \Omega^k(M)$ we have $\iota_X(\beta)(X_1, ..., X_{k-1}) = \beta(X, X_1, ..., X_{k-1})$

$$\mathcal{L}_X(\omega) = \iota_X(d\omega) + d(\iota_X \omega)$$

Definition: (Integration)
Let M be a smooth n-manifold with $K \subseteq M$ a smooth k-dimensional sub-manifold and $\omega \in \Omega^k(M)$. We then have $\iota : K \hookrightarrow M$. Define

$$\int_{\iota(K)} \omega = \int_K \iota^*(\omega)$$

Let $\{(U_i, \varphi_i : \Omega_i \to U_i)\}_{i \in I}$ be an atlas in the smooth structure for M and $1 = \sum_{i \in I} \rho_i$ be a partition of unity where $\rho_i \in C^\infty(M)$ and $supp(\rho_i) \subseteq U_i$

(For details on existence and properties see the last section of chapter 2 of [Spi70] or the 3rd appendix in [KN63])

Noting that there is a function $f_i \in C^\infty(\Omega_i)$ such that $(\rho_i \circ \varphi_1)(\iota \circ \varphi_i)^*(\omega) = f_i dx_1 \wedge ... \wedge dx_k$ where x_j are the standard coordinate functions in $\mathbb{R}^k$ we define

$$\int_K \iota^*(\omega) = \sum_{i \in I} \int_{\Omega_i} \rho_i(\iota \circ \varphi_i)^*(\omega) = \sum_{i \in I} \int_{\Omega_i} f_i dx_1 ... dx_k$$

where the elements of the sum are given by integration (Lebesgue or Riemannian) in $\mathbb{R}^k$. If the sum is not defined then we say the integral is not defined. This is independent from the choice of atlas and partition of unity.

Definition: (Riemannian Metric)
Let M be a smooth manifold. Let $\eta \in \Gamma(T^*M \otimes T^*M)$. If for all $x \in M$ we have $\eta_x : T_xM \otimes T_xM \to \mathbb{R}$ is a non-degenerate (or positive definite) bilinear form then we say that η is a **Riemannian metric** on M and (M, η) is a **Riemannian manifold**.

Definition: (Hodge-$\star$ operator)
Let (M, η) be a Riemannian n-manifold. Then the Hodge-$\star$ operator $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ is uniquely determined such that for $\alpha, \beta \in \Omega^k(M)$ we have

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle_\eta Vol(\eta)$$

where $Vol(\eta)$ is the volume form induced by η and $\langle -, - \rangle_\eta : \Omega^k(M) \otimes \Omega^k(M) \to C^\infty(M)$ is the non-degenerate pairing defined by η.

Definition: (The Fundamental Group)
Let M be a manifold. Let $\pi_1(M, x) = \{\gamma : [0,1] \to M : \gamma(0) = \gamma(1) = x\}/\sim$ where $\gamma_1 \sim \gamma_2$ if there exists $H : [0,1] \times [0,1] \to M$ such that $H(t, 0) = \gamma_1(t)$ and $H(t, 1) = \gamma_1$ and $H(0, s) = H(1, s) = x$.

This set can be given a group structure. For $[\gamma_1], [\gamma_2] \in \pi_1(M)$ we take representatives γ_1, γ_2 and let

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Then define $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$. This is well defined. We can see that for $[\gamma] \in \pi_1(M, x)$ and $\overline{\gamma} : [0,1] \to M$ such that $\overline{\gamma}(t) = \gamma(1-t)$ we have

$$[\overline{\gamma}] = [\gamma]^{-1}$$

See [Hat02] for details and results concerning the fundamental group.

Appendix B

Lie Groups, Lie Algebras and Affine Lie Algebras

We briefly define Lie groups and Lie algebras and describe their correspondence. We then define the affine Lie algebras and discuss some of the irreducible modules of the Lie algebras and the affine Lie algebras. For a detailed exhibition of the theory surrounding Lie algebras and lie groups see [Bou98] or [Hum12]. For some of the aspect related to the Fourier analysis on Lie groups see [Bt03]. For a detailed exhibition of the theory of affine Lie algebras see [Kac94] however an extremely concise reference that covers the important points needed for this thesis is [Bea94].

B.1 Lie Groups and Associated Lie Algebra

Definition: (Lie Group)
If G is a smooth manifold with the structure of a group with multiplication given by $\cdot$ such that the group multiplication $\cdot : G \times G \to G$ and inversion $-^{-1} : G \to G$ are smooth then we say G is a **Lie group**.

Definition: (Lie Algebra)
Let $\mathfrak{g}$ be a vector space with an bilinear antisymmetric bracket operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for $u, v, w \in \mathfrak{g}$ we have

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

This last condition is called the Jacobi identity. $(\mathfrak{g}, [-,-])$ is called a **Lie algebra**.

Example: (Lie Algebra of Vector Fields)
Let M be a smooth manifold. Let $\mathfrak{X}(M) = \Gamma(TM)$ be the space of vector fields. Consider $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. We then get a smooth functions $X(f) \in C^\infty(M)$ such that for $x \in M$ we have $X(f)(x) = X_x(f)$. Define the following bracket operation $[-,-] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

$$[X, Y]_x(f) = X_x(Y(f)) - Y_x(X(f))$$

It can be shown that $\mathcal{L}_X(Y) = [X, Y]$ where $\mathcal{L}_X$ is the Lie derivative in the direction of X.

Definition: (Lie Algebra Associated to a Lie Group)
Let G be a Lie group and $e \in G$ the identity in G. For $g \in G$ define $R_g : G \to G$ to be $R_g(h) = h \cdot g$. For $X_e \in T_eG$ define the vector field X such that $X_g = T_eR_g(X_e)$. Let $Lie(G) = \mathfrak{g}$ and define the bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that for $X_e, Y_e \in \mathfrak{g}$ we have

$$[X_e, Y_e] = [X, Y]_e$$

where the right hand side is given by the Lie bracket of vector fields.

Definition: (The Exponential Map)
Let G be a Lie group and $Lie(G) = \mathfrak{g}$ be the associated Lie algebra. We can define a map $\exp : \mathfrak{g} \to G$ as follows. Let $X_e \in \mathfrak{g}$ and take the vector field $X \in \mathfrak{X}(G)$ such that $X_g = T_eR_g(X_e)$. Take the flow of X through e given by $\gamma : \mathbb{R} \to G$. Define $\exp(X) = \gamma(1)$.

Lemma B.1.1. *Let G be a Lie group and $Lie(G) = \mathfrak{g}$ be the associated Lie algebra. Let $X \in \mathfrak{g}$. The map $\exp_X : \mathbb{R} \to G$ such that $\exp_X(t) = \exp(tX)$ defines a Lie group homomorphism from $(\mathbb{R}, +)$ to G.*

Remark: *Lie* is a functor from Lie groups to Lie algebras. With enough adjectives this functor is an equivalence of categories with the exponential map giving the quasi-inverse functor.

B.2 Representations, Functions on Lie Groups and Integration

Definition: (Representation)
Let G be a Lie group. A **representation** of G is a group homomorphism for some vector space V given by $\rho : G \to End(V)$

Remark: Given a G-representation we can define V as a G module and visa versa.

Definition: (Irreducible Representation)
A G-representation is **irreducible** if the G-module associated to the representation has no proper sub-modules (i.e is simple).

Definition: (Class functions)
Let G be a Lie group. A **class function** is a function $f : G \to \mathbb{C}$ such that $f(g) = f(hgh^{-1})$ for all $g, h \in G$.

Definition: (Character of a G-representation)
Let G be a Lie group. The **character** of a representation $\rho : G \to End(V)$ is given by $\chi_\rho = Tr \circ \rho : G \to \mathbb{C}$.

Definition: (Adjoint Actions)
Let G be a Lie group and $\mathfrak{g} = Lie(G)$ be the corresponding Lie algebra. Define the adjoint action of G on $\mathfrak{g}$ as follows for $g \in G$, $X \in \mathfrak{g}$ and $C_g : G \to G$ such that $C_g(h) = ghg^{-1}$.

$$Ad_g(X) = T_e C_g(X)$$

Now define the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ as follows for $X, Y \in \mathfrak{g}$

$$ad_X(Y) = [X, Y]$$

Remark: The adjoint action defines a representation of G into $End(\mathfrak{g})$.

Definition: (The Killing Form)
Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following form on $\mathfrak{g}$ is called the **Killing form**. For $X, Y \in \mathfrak{g}$ we have

$$\kappa(X, Y) = \langle X, Y \rangle = Tr(ad_X \circ ad_Y)$$

This is symmetric from properties of the trace function and is non-degenerate if and only if $\mathfrak{g}$ is semi-simple (a direct product of Lie algebras with only trivial ideals).

Definition: (Maurer-Cartan Form)
Let G be a Lie group. Define the **Maurer-Cartan** form $\omega_G \in \Omega^1(G, \mathfrak{g})$ as follows

$$(\omega_G)_g(X_g) = T_g L_{g^{-1}}(X_g)$$

Lemma B.2.1. *Let G be a Lie group and ω_G the Maurer-Cartan form. For $v, u \in T_gG$*

$$(d\omega_G)_g(u, v) + [(\omega_G)_g(v), (\omega_G)_g(u)] = 0$$

In the notation developed in section D.1 we have

$$d\omega_G + [\omega_G \wedge \omega_G] = 0$$

Proof. Let $u, v \in T_gG$. Consider $X, Y \in \mathfrak{X}(G)$ such that $X_h = T_g L_{hg^{-1}}(u)$ and $Y_h = T_g L_{hg^{-1}}(v)$. Notice that $X_g = u$ and $Y_g = v$. Then note that $\omega_G(X) = T_g L_{g^{-1}}(u)$ and $\omega_G(Y) = T_g L_{g^{-1}}(v)$. So

$$(d\omega_G)_g(u, v) = (d\omega_G)_g(X_g, Y_g) = u(\omega_G(Y)) - v(\omega_G(X)) - (\omega_G)_g([X, Y]_g)$$

$$= -(\omega_G)_g([X, Y]_g) = -T_g L_{g^{-1}}([X, Y]_g) = [-T_g L_{g^{-1}}(X_g), -T_g L_{g^{-1}}(Y_g)]$$
$$= -[(\omega_G)_g(v), (\omega_G)_g(u)]$$

□

Definition: (Invariant Metric)
Let G be a Lie group, κ a killing form and ω_G the Maurer-Cartan form. We can define a Riemannian metric on G we'll denote $\gamma \in \Gamma(T^*G \otimes T^*G)$ such that for $g \in G$ and $X_g, Y_g \in T_gG$ we have

$$\gamma_g(X_g, Y_g) = \kappa(\omega_g(X_g), \omega_g(Y_g))$$

Remark: For $\mathfrak{g}$ semi simple the invariant metric defines above defines a volume form on G which induces a measure on G.

Definition: (Haar Measure)
Let G be a Lie group. There exists a unique Borel measure dg on G called the **Haar measure** that satisfies

- For some measurable set S we have $\int_S dg = \int_{gS} dg$
- $\int_G dg = 1$

Remark: Haar measures can be defined on any topological vectors space and smoothness isn't needed in the construction.

Definition: ($L^2(G)$)
Let G be a Lie group. Let $L^2(G) = \{f : G \to \mathbb{C} : \int_G f(g)^2 dg < \infty\}$ be the vector space of **square integrable functions** on G with norm given by $\|f\| = \left(\int_G f(g)^2 dg\right)^{\frac{1}{2}}$.

Definition: (Representative Functions)
Let G be a Lie group. Let $f : G \to \mathbb{C}$ be a continuous function. We say that f is **representative** if $\{h : G \to \mathbb{C}$: there exists $g \in G$ such that $h(x) = f(xg)\}$ is finite dimensional.

Remark: These representative functions are given by composing a representation $\rho : G \to End(V)$ and a linear map $F : End(V) \to \mathbb{C}$.

Definition: (Banach Algebra)
Let G be a compact Lie group. Then $C^0(G, \mathbb{C})$ is an Banach algebra with the supremum norm $|f| = \sup_{g \in G} |f(g)|$ and product

$$(f * g)(x) = \int_G f(z) g(z^{-1}x) dz$$

Theorem B.2.2. *(Peter-Weyl Theorem)(See chapter III of [Bt03])*
Let G be a compact Lie group.

- *The representative functions are dense in both $C^0(G, \mathbb{C})$ and $L^2(G)$.*
- *The irreducible characters generate a dense subspace of the space of continuous class functions.*

Definition: (Torus)
Let G be a compact Lie group. A **torus** in G is a closed subgroup T such that T is connected and Abelian. Note that the only connected compact Abelian Lie groups are given by $U(1)^n$. We say that T is **maximal** if for another torus T' we have $T \leqslant T' \leqslant G$ then $T = T'$.

Definition: (Cartan Sub-algebra)
Let G be a compact connected Lie group and $\mathfrak{g} = Lie(G)$. Let $T \leqslant G$ be a maximal torus. Then $Lie(T) = \mathfrak{h} \subseteq \mathfrak{g}$ is called a **Cartan Sub-algebra**.

Definition: (Weights)
Let G be a compact connected Lie group. Let V be a representation. Let $h \in \mathfrak{h}$ and $v \in V$ such that $hv = \lambda(h)v$ for some $\lambda(h) \in \mathbb{C}$. Now as $\mathfrak{h}$ is Abelian every $k \in \mathfrak{h}$ acts via a constant $kv = \lambda(k)v$. This defines an element $\lambda \in \mathfrak{h}^*$. We say that v is a **weight vector** of **weight** $\lambda \in \mathfrak{h}^*$.

Theorem B.2.3. *(Decomposition into Roots)*
Let G be a simply connected, compact, semi-simple Lie group. Then we have the following decomposition for $R = \{\lambda \in \mathfrak{h}^* : [h, X] = \lambda(h)X\}$.

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha)$$

Remark: For semi-simple compact $\mathfrak{g}$ the set R can be given be embedded in Euclidean space. The arrangements of R in Euclidean space satisfy certain properties that make R what is called a root system. Classifying all root systems then classifies all semi-simple Lie algebras.

Theorem B.2.4. *(Classification of Semi-simple Lie Algebras)*
All finite dimensional simple (only trivial ideals) complex Lie algebras are isomorphic to one of the following for $n \in \mathbb{Z}_{>0}$

$$A_n = \mathfrak{sl}_{n+1}(\mathbb{C}), \quad B_n = \mathfrak{so}_{2n+1}(\mathbb{C}), \quad C_n = \mathfrak{sp}_{2n}(\mathbb{C}), \quad D_n = \mathfrak{so}_{2n}(\mathbb{C}), \quad E_6, E_7, E_8, F_4, G_2$$

where E_6, E_7, E_8, F_4, G_2 are called the exceptional simple complex Lie algebras. We won't define them here for definitions see [Hum12]. The letters A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 correspond to the irreducible root systems. Again see [Hum12] for definition of a root system and how they are associated to Lie algebras.

Theorem B.2.5. *(Classification of Irreducible Representations)*
For a simple complex Lie algebra $\mathfrak{g}$ there is an associated root system say R. Let P be the integer lattice spanned by the positive integral weights of R. The finite irreducible $\mathfrak{g}$-modules are in bijection with P. They are explicitly constructed as maximal quotients of Verma modules associated to the elements of P. See [Hum12] for details.

Remark: We can translate the previous two statements into statements about Lie groups using the association between Lie groups and Lie algebras via the exponential map and taking the left invariant vector fields.

Theorem B.2.6. *(Weyl Integration Formula)(See Chapter IV of [Bt03])*
Let G be a compact connected Lie group, $T \leqslant G$ a maximal torus, $W = N(T)/T$ the Weyl group, $\mathfrak{g} = Lie(G)$, $\mathfrak{h} = Lie(T)$ and $f \in C^0(G, \mathbb{C})$. Letting $\det\left(I - ad_{\mathfrak{g}/\mathfrak{h}}(t^{-1})\right) = F(t)$ we have

$$|W| \int_G f(g)dg = \int_T \left(F(t) \int_G f(gtg^{-1})dg\right) dt$$

Corollary B.2.7. *Noting that $\int_G dg = 1$ if f is a class function then*

$$|W| \int_G f(g)dg = \int_T f(t)F(t)dt$$

Theorem B.2.8. *(Weyl Denominator Theorem)*
Let G be a compact connected Lie group with maximal torus T and Weyl group $W = N(T)/T$. Let $\mathfrak{h} = Lie(T) \subseteq \mathfrak{g} = Lie(G)$. Let $P \subseteq \mathfrak{h}^*$ be the set of positive roots and $\rho = \frac{1}{2}\sum_{\lambda \in P} \lambda$. Let $h \in \mathfrak{h}$ then consider $\exp(h) \in T$. Then we have

$$F(\exp(h)) = \sum_{w \in W} sign(w) e^{\rho(w \cdot h)}$$

Theorem B.2.9. *(Weyl Character Formula)*
Let G be a compact connected Lie group with maximal torus T and Weyl group $W = N(T)/T$. Let $\mathfrak{h} = Lie(T) \subseteq \mathfrak{g} = Lie(G)$. Let $P \subseteq \mathfrak{h}^*$ be the set of positive roots and $\rho = \frac{1}{2}\sum_{\lambda \in P} \lambda$. Let $h \in \mathfrak{h}$ then consider $\exp(h) \in T$. Let V be an irreducible representation of G with highest weight $\lambda \in \mathfrak{h}^*$ then

$$\chi_V(\exp(h)) = \frac{\sum_{w \in W} sign(w) e^{(\lambda+\rho)(w \cdot h)}}{F(\exp(h))} = \frac{\sum_{w \in W} sign(w) e^{(\lambda+\rho)(w \cdot h)}}{\sum_{w \in W} sign(w) e^{\rho(w \cdot h)}}$$

Corollary B.2.10. *(Weyl Dimension Formula)*
By taking the limit as $h \to 0$ and applying L'Hôpital's rule we can show that

$$\dim(V) = \frac{\prod_{\alpha \in P} \kappa^*(\alpha, \lambda + \rho)}{\prod_{\alpha \in P} \kappa^*(\alpha, \rho)}$$

B.3 Affine Lie Algebras

Definition: (Affine Lie Algebra)
Let $\mathfrak{g}$ be a simple complex Lie algebra. Define the affine Lie algebra associated to $\mathfrak{g}$ to be a central extension of the Lie algebra tensored with formal Laurent series $\widehat{\mathfrak{g}} = (\mathbb{C}((z)) \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}c$ with c central and the bracket given by

$$[f(z) \otimes X, g(z) \otimes Y] = f(z)g(z) \otimes [X, Y] + \langle X, Y \rangle Res_{z=0}(gdf)c$$

Note that $0 = Res_{z=0}d(fg) = Res_{z=0}(fdg + gdf)$ so this antisymmetric.

Definition: (Level)
Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and V a $\widehat{\mathfrak{g}}$-module. If $c \in \widehat{\mathfrak{g}}$ is the element of the central extension then for some $l \in \mathbb{C}$ and for all $v \in V$ we have $c \cdot v = lv$. We call l the **level** of the representation.

Theorem B.3.1. *Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and $l \in \mathbb{Z}_{>0}$. Let P be the positive integral weights associated to $\mathfrak{g}$. The reasonable irreducible representations of $\widehat{\mathfrak{g}}$ of level l are in bijection with P_l. Where P_l is the set of positive integral weights such that evaluating on the highest root is less than equal to l.*

Appendix C

The Theory of Principle Bundles and their Connections

We will define and give some immediate results relating to principle G-bundles and their connections. We will touch on all the details we will need to define the moduli space of flat connections. A good introduction to this is provided in [KN63] and [Mor01].

C.1 Principle Bundles

Definition: (Principle Bundle)
Let G be a Lie group and $(\pi : P \to M, F)$ a smooth fibre bundle. If G acts smoothly on P via $\cdot : P \times G \to P$ such that for $p \in P$ and $g \in G$

$$\pi(p \cdot g) = \pi(p)$$

and G acts freely and transitively on $\pi^{-1}(x)$ for all $x \in M$ then we say $(\pi : P \to M, F, G, \cdot)$ is a **principle G-bundle**. Notice that $M \cong P/G$ and $F \cong G$. So we can write our principle bundle as $(\pi : P \to M, G, \cdot)$. We take the morphisms between principle G-bundles to be equivariant smooth bundle maps.

Example: (Principle Bundle)

- Let M be a smooth manifold, G a Lie Group, $\varphi_M : M \times G \to M$ such that $\varphi_M(x, g) = x$ and for $(x, g) \in P$ and $h \in G$ we have $(x, g) \cdot h = (x, gh)$. We call $(\varphi_M : M \times G \to M, G, \cdot)$ the trivial G-bundle over M. Notice that for $M = \textit{point}$ the theory of principle bundles is the same as the theory of Lie groups.
- Let $\pi : \mathcal{L} \to M$ be a complex line bundle with a norm ν. Let $P = \{p \in \mathcal{L} : \nu(p) = 1\}$ and $\pi|_P : P \to M$. Then we can see that $\pi|_P^{-1}(x) \cong U(1)$ and that $U(1)$ acts naturally on $\pi|_P^{-1}(x)$ via the (scalar) complex multiplication defined by $\mathcal{L}$ and that this action is free and transitive on the fibres. It can be seen that $\pi|_P : P \to M$ is a fibre bundle and so defines a principle bundle.

More explicitly take $M = \mathbb{CP}^n$, $P = \{(l, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in l \text{ and } \|z\| = 1\}$, $G = U(1)$ and for $(l, z) \in P$ and $e^{i\theta} \in G$ have $(l, z) \cdot e^{i\theta} = (l, e^{i\theta} z)$.

Notation: We will often shorten the notation $(\pi : P \to M, G, \cdot)$ to simply P if the additional structure is clear from context.

Remark: Note that a manifold with a free proper G-action defines a principle bundle with total space the manifold and base space the orbit space which inherits a natural smoothness structure. To prove this one needs a result such as the slice theorem. See [Aud04] or [dS02].

Remark: If we instead took G to be a finite group in our definition then we would be considering covering spaces with Deck transformations given by G. In fact it is sometimes possible to change the topology of P to make it into a covering space however there is no canonical way to do this in general. To do this canonically we need more structure such as a flat connection. In this sense all connections for a finite group are flat.

Remark: Every n-dimensional $\mathbb{F}$-vector bundle $(\nu : E \to B, \mathbb{F}^n)$ can be realised as by a principle $GL_n(\mathbb{F})$-bundle $(\pi : P \to M, GL_n(\mathbb{F}), \cdot)$ as follows

$$E = P \times \mathbb{F}^n / \sim \quad \text{and} \quad \nu : E \to B = M \quad s.t \quad \nu(p, v) = \pi(p)$$

where for $p \in P$, $v \in \mathbb{F}^n$ and $A \in GL_n(\mathbb{F})$ we have $(p \cdot A, v) \sim (p, A(v))$ where we view $GL_n(\mathbb{F}) = Aut(\mathbb{F}^n)$. We get the structure of a vector space in each fibre from the vector space structure of $\mathbb{F}^n$.

Lemma C.1.1. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. If there exists a smooth map $s : M \to P$ such that $\pi \circ s = id$ then P is a trivial bundle.*

Proof. Define $\Phi : M \times G \to P$ such that $\Phi(x, g) = s(x) \cdot g$ and check that this is indeed an isomorphism of principle bundles. □

To each principle bundle we can define a canonical vector bundle as follows.

Definition: (Adjoint Bundle)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $\mathfrak{g}$ be the Lie algebra of G. Let G act on $\mathfrak{g}$ by the left adjoint action. Define the following bundle over M

$$\pi_P : \mathfrak{g}_P = P \times_G \mathfrak{g} \to M \quad s.t \quad \pi_P(p, v) = \pi(p)$$

The notation $\times_G$ means that we impose the following equivalence relation on $P \times \mathfrak{g}$

$$(p \cdot g, v) \sim (p, Ad_g(v))$$

where we have taken the left adjoint action. Notice that this is well defined as $\pi(p \cdot g) = \pi(p)$. We call this vector bundle the **adjoint $\mathfrak{g}$-bundle** associated to P.

Many structures on P can be determined by structures on $\mathfrak{g}_P$.

Canonical functions and G action on TP: Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. We have the following canonical smooth functions

- For $g \in G$ define $R_g : P \to P$ such that $R_g(p) = p \cdot g$
- For $p \in P$ define $L_p : G \to P$ such that $L_p(g) = p \cdot g$
- For $p \in P$ define $I_p : \pi^{-1}(\pi(p)) \to G$ such that $I_p(p \cdot g) = g$.
- For $g \in G$ define $R_{g,P} : \mathfrak{g}_P \to \mathfrak{g}_P$ such that $R_{g,P}(p, v) = (p \cdot g, v) \sim (p, Ad_g(v))$
- For $(v, p) \in \mathfrak{g}_P$ define $L_{(p,v)} : G \to \mathfrak{g}_P$ such that $L_{(p,v)}(g) = (p \cdot g, v) \sim (p, Ad_g(v))$

Given a principle bundle $(\pi : P \to M, G, \cdot)$ we will take the these functions above with this notation as a given.

As G acts freely and transitively on the fibres of P we can see that L_p defines an equivariant diffeomorphism onto the fibre containing p (if we take right group multiplication as the action of G on itself) with inverse given by I_p.

Given an element $v \in \mathfrak{g}$ we can define a vector field of P which can be geometrically interpreted as the direction of the infinitesimal action of v of P. This is called the **fundamental vector field** associated to v. We will denote it v^* and it is defined as $v_p^* = T_e L_p(v)$.

We can use the functions R_g to define a natural action of G on TP. For $q \in P$ we can see that by the chain rule $\cdot : TP \times G \to TP$ such that $v_q \cdot g = T_q R_g(v_q)$ defines an action of G on TP and this will vary smoothly with $g \in G$ so in fact defines a smooth action on TP.

C.2 Connections on Principle Bundles

Definition: (Connection)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $H \subseteq TP$ be a smooth sub-bundle such that for $V = ker(\pi_*)$ we have $TP = V \oplus H$ and $T_p R_g(H_p) = H_{p \cdot g}$ for all $g \in G$ and $p \in P$. We call H a **connection** on P.

Remark: We call V the vertical tangent space and H the horizontal tangent space of the connection H.

Example: (Connections)

- Let $\pi : M \times G \to M$ be the trivial bundle. Then let $H = \{(X_p, 0) \in TM \times TG\}$. This is called the trivial connection on a trivial bundle.

- Let $\pi : M \times G \to M$ be the trivial G-bundle over M. Let $f : M \to G$ be smooth and let $H = \{(X_x, T_x(R_g \circ f)(X_x)) \in TM \times TG : x \in M, g \in G\}$. Notice that by Lemma 1.1 this is in fact isomorphic to the trivial bundle with trivial connection.

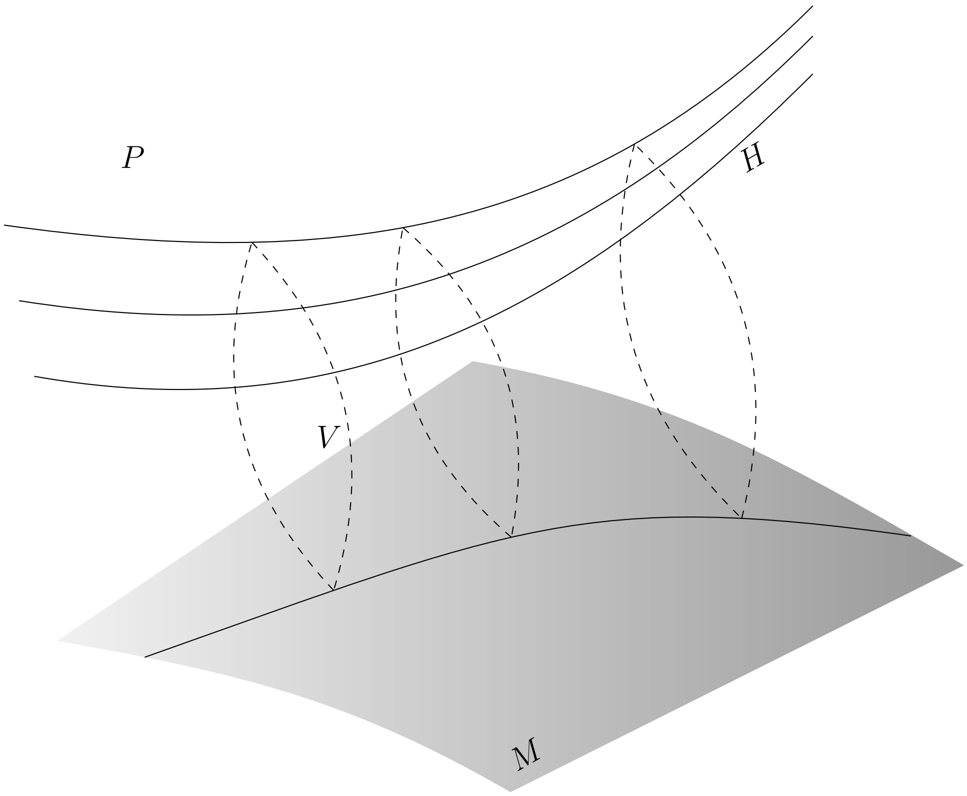
Definition: (Principle Bundle with Connection)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and H a connection on P. We call $(\pi : P \to M, G, \cdot, H)$ a **principle bundle with connection**. We will often denote it simply (P, H).

This forms a category where for (P, H_1) and (Q, H_2) principle bundles over G with connections we take the morphisms as morphisms of principle bundles

$$f : P \to Q \quad s.t \quad Tf|_{H_1} : H_1 \to H_2$$

Note that taking bundle maps means vertical tangent vectors are sent to vertical tangent vectors and the other condition on f says that horizontal tangent vectors are sent to horizontal tangent vectors.

Example: We have depicted a local piece of a principle $U(1)$-bundle $\pi : P \to M$. As always our vertical tangent space V is naturally given to us by the bundle via $V = ker(d\pi)$ and our horizontal tangent space H is given to us by our connection.



Definition: (Lie-Algebra Valued k-forms)
Let M be a smooth manifold and G a Lie group with Lie algebra $Lie(G) = \mathfrak{g}$. The vector space $\Omega^k(M, \mathfrak{g}) = \Gamma\left((M \times \mathfrak{g}) \otimes (\Lambda^k T^* M)\right) \cong \Omega^k(M) \otimes \mathfrak{g}$

is the space of **Lie-Algebra Valued k-Forms** on M. Notice that these are sections of the bundle of antisymmetric k-linear homomorphisms from the tangent space to $\mathfrak{g}$.

Definition: (Pseudotensorial Forms, Tensorial Forms and Connection 1-forms)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. We say that $\beta \in \Omega^k(P, \mathfrak{g})$ is **pseudotensorial** if for $g \in G$ we have $R_g^*(\beta) = \beta \circ TR_g = Ad_{g^{-1}} \circ \beta$.

We say β is **tensorial** if it is pseudotensorial and if for some $i \in \{1, ..., k\}$ we have $X_i \in V \subseteq TP$, the vertical tangent vectors, then $\beta(X_1, ..., X_k) = 0$.

We say that $\beta \in \Omega^1(P, \mathfrak{g})$ is a **connection 1-form** if β is pseudotensorial and $\beta_p(T_e L_p(v)) = v$, that is for the fundamental vector field associated to v given by v^* we have $\beta(v^*) = v$.

Definition: (k-forms with Values in the Adjoint Bundle)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Define $\Omega^k(M, \mathfrak{g}_P) = \Gamma(\Lambda^k T^* M \otimes \mathfrak{g}_P)$ to be the space of **k-forms with Values in the Adjoint Bundle**.

Lemma C.2.1. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $\beta \in \Omega^k(P, \mathfrak{g})$ be tensorial. There is a canonical k-form $\beta_P \in \Omega^k(M, \mathfrak{g}_P)$ associated to β. Moreover this association is bijective.*

Proof. Given a tensorial k-form $\beta \in \Omega^k(P, \mathfrak{g})$ define $\beta_P \in \Omega^k(M, \mathfrak{g}_P)$ such that for $X_x \in (T_x M)^{\otimes k}$ take $Y_p \in (T_p^{\otimes k}\pi)^{-1}(X_x)$ and let $\beta_P(X_x) = (p, \beta_p(Y_p))$ and this will be well defined as β is tensorial.

Given $\beta_P \in \Omega^k(M, \mathfrak{g}_P)$ define $\beta \in \Omega^k(P, \mathfrak{g})$ to be the form such that if $\beta_P(T_p^{\otimes k}\pi(Y_p)) = (p, Z_p)$ we have $\beta(Y_p) = Z_p$. This will be tensorial. $\square$

There are multiple ways to view a connection. Here we list a few.

Lemma C.2.2. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. The following objects are canonically bijective*

- *Connections on P.*
- *Smooth linear equivariant vertical projections $\varphi_V : TP \to V$ that is a splitting of the short exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, hook] & TP \arrow[r, two heads] \arrow[l, dashed, bend left, "\varphi_V"] & TP/V \arrow[r] \arrow[l, dashed, bend left, hook] & 0 \end{tikzcd}$$

- *Connection 1-forms on P*

Proof. Firstly we clarify what is meant by smooth linear equivariant vertical projections $\varphi_V : TP \to V$. That is for $p \in P$ a linear projection $(\varphi_V)_p : T_pP \to V_p$ that vary smoothly with p and is such that $(\varphi_V)_{p\cdot g} \circ T_pR_g = T_pR_g \circ (\varphi_V)_p$.

Given a connection H for $X_p \in T_pP$ we have unique $X_{V_p} \in V_p$ and $X_{H_p} \in H_p$ such that $X_p = X_{V_p} + X_{H_p}$. Let $(\varphi_V)_p(X_p) = X_{V_p}$. This will be smooth as we vary $p \in P$. It is a linear projection of T_pP onto V_p. It is equivariant as $T_pR_g(H_p) = H_{p\cdot g}$.

Given a smooth linear and equivariant projection $\varphi_V : TP \to V$ define $H = ker(\varphi_V)$.

The map R_p defines an isomorphism between G and the fibre of P containing p. Therefore we see that $T_eL_p : T_eG = \mathfrak{g} \to V_p$ is an isomorphism. We use this isomorphism to pass between smooth linear and equivariant vertical projections and connection 1-forms.

If $\beta \in \Omega^1(P, \mathfrak{g})$ is associated to φ_V then the equivariance of becomes φ_V becomes

$$T_{p\cdot g}I_{p\cdot g} \circ (\varphi_V)_{p\cdot g} \circ T_pR_g = T_{p\cdot g}I_{p\cdot g} \circ T_pR_g \circ (\varphi_V)_p$$

$$\Rightarrow \beta_{p\cdot g} \circ T_pR_g = T_{p\cdot g}I_{p\cdot g} \circ T_pR_g \circ T_eL_p \circ \beta_p = Ad_{g^{-1}} \circ \beta_p$$

As for $h \in G$ we have $I_{p\cdot g} \circ R_g \circ L_p(h) = I_{p\cdot g} \circ R_g(p \cdot h) = I_{p\cdot g}(p \cdot hg) = I_{p\cdot g}(p \cdot gg^{-1}hg) = g^{-1}hg$ and by the chain rule and definition of the adjoint action $Ad_{g^{-1}} = T_e\left(I_{p\cdot g} \circ R_g \circ L_p\right) = T_{p\cdot g}I_{p\cdot g} \circ T_pR_g \circ T_eL_p$. Now the projection property of the smooth linear equivariant vertical projections becomes the following

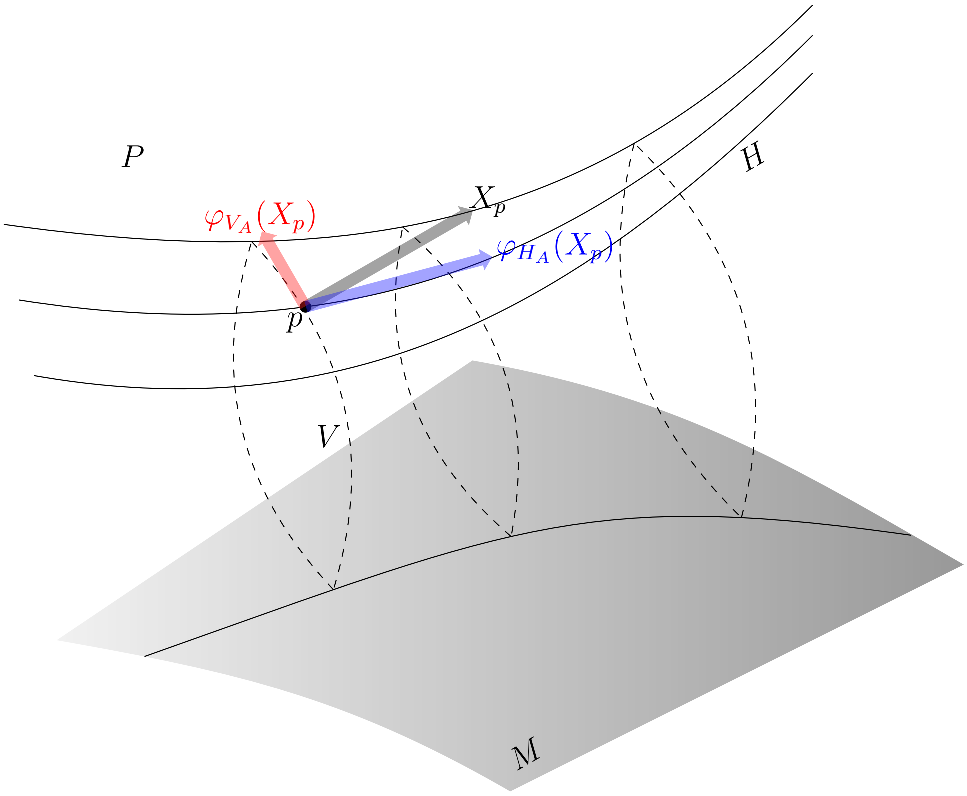
$$(\varphi_V)_p(T_eL_p(v)) = T_eL_p \circ \beta_p(T_eL_p(v)) = T_eL_p(v)$$

These two conditions exactly state that β is a connection 1-form. □

Remark: The objects in bijection with the connections on P described in lemma C.2.2 have natural topologies and in fact are homeomorphic with respect to the bijection.

Remark: The condition that says for a connection 1-form β, $v \in \mathfrak{g}$ and $p \in P$ that $\beta_p(T_eL_p(v)) = v$ or in other words $\beta(v^*) = v$ says that $\beta|_{\pi^{-1}(\pi(p))}$ is the Maurer-Cartan form on $\pi^{-1}\left(\pi(p)\right)$ using L_p to identify G and $\pi^{-1}\left(\pi(p)\right)$.

Remark: This theorem geometrically says that projections determine the horizontal space. We have again depicted a principle $U(1)$-bundle $\pi : P \to M$. We have let $\varphi_{V_A} : TP \to V$ and $\varphi_{H_A} : TP \to H$ be vertical and horizontal projections associated to a connection A.



Locally principle bundles are trivial and therefore connections on trivial principle bundles are fundamental. We have the following important result.

Lemma C.2.3. *Let $P = M \times G$ be the trivial principle G bundle over M. The connections on P are in canonical bijection with $\Omega^1(M, \mathfrak{g}) \cong \Omega^1(M, \mathfrak{g}_{M\times G})$.*

Proof. Notice that for a 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ we can define a connection 1-form as $\beta \in \Omega^1(P, \mathfrak{g})$ such that for $(u, v) \in T_xM \times T_gG$ we have $\beta_{(x,g)}(u, v) = Ad_{g^{-1}}(\alpha_x(u)) + (\omega_G)_g(v)$ where ω_G is the Maurer-Cartan form. Similarly we can given a connection 1-form $\beta \in \Omega^1(P, \mathfrak{g})$ we can define a 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$ such that for $v \in T_xM$ we have $\alpha_x(v) = \beta_{(x,e)}(v, 0)$. These correspondences give the canonical bijection. $\square$

> **Remark:** We need a section of the manifold for this argument to go through. So this only works for the trivial bundle. The canonical part of the theorem comes from the canonical trivial section which is sent to $0 \in \Omega^1(M, \mathfrak{g})$.

C.3 Space of Connections

Affine Structure: Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Both the smooth linear equivariant vertical projections from TP to V and the connection 1-forms are subsets of spaces with a natural addition and in fact the structure of a vector space. We claim that the difference inherited from these spaces gives the smooth linear equivariant vertical projections from TP to V and the connection 1-forms the structure of an affine space.

Lemma C.3.1. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. The difference inherited by $\Omega^1(P, \mathfrak{g})$ gives the connection 1-forms on P the structure of an affine space based on $\Omega^1(M, \mathfrak{g}_P)$.*

Proof. Let α and β be two connection 1-forms on P. Notice that for $p \in P$ and $X \in V_p$ we have

$$(\alpha - \beta)_p(X_p) = \alpha_p(X_p) - \beta_p(X_p) = T_pI_p(X_p) - T_pI_p(X_p) = 0$$

Notice that as α and β are connection 1-forms we see that $(\alpha - \beta)_{p \cdot g} \circ T_pR_g(X_p) = Ad_{g^{-1}} \circ (\alpha - \beta)_p(X_p)$. This shows that $(\alpha - \beta)$ is a tensorial 1-form. From lemma C.2.1 we can see that there is a canonical 1-form in $\Omega^1(M, \mathfrak{g}_P)$ associated to $(\alpha - \beta)$.

Notice that if for $\gamma \in \Omega^1(M, \mathfrak{g}_P)$ and for $X_p \in T_pP$ we have that $\gamma_{\pi(p)}(T_p\pi(X_p)) = (p, Y_p)$ then $(\alpha + \gamma)_p(X_p) = \alpha_p(X_p) + Y_p$ defines a connection 1-form on P.

Therefore the connection 1-forms on P have the structure of an affine space with associated vector space $\Omega^1(M, \mathfrak{g}_P)$. □

> **Definition:** (Space of Connections)
> Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Define $\mathcal{A}_P$ to be the **affine space of connections on** P with the topology inherited from $\Omega^1(M, \mathfrak{g}_P)$.
>
> **Remark:** Notice that for trivial bundle $\mathcal{A}_{M \times G} \cong \Omega^1(M, \mathfrak{g}) \cong \Omega^1(M, \mathfrak{g}_{M \times G})$ is in fact a vector space with the 0 vector given by the trivial connection. For the general bundle there is no such canonical 0 so we only get an affine space.
>
> **Remark:** In fact $\Omega^1(M, \mathfrak{g}_P)$ can be given a smooth structure which means that $\mathcal{A}_P$ can be given a smooth structure. Notice this means we need to define what a smoothness structure on an infinite dimensional manifold is. For those interested see [KM97] for some of the theory needed.

Tangent Space: The tangent space at a point of a smooth affine space is isomorphic to the vector space the affine space is based on. This means that for $A \in \mathcal{A}_P$ we have $T_A\mathcal{A}_P \cong \Omega^1(M, \mathfrak{g}_P)$.

Notation: Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. We will use the following notation

- $H_A \subseteq V \oplus H_A = TP$ will be the horizontal subspace associated to A
- φ_{V_A} will be the equivariant vertical projection associated to A.
- $id - \varphi_{V_A} = \varphi_{H_A}$ will be the equivariant horizontal projection associated to A
- $\omega_A \in \Omega^1(P, \mathfrak{g})$ will be the connection 1-form associated to A

Appendix D

Additional Structures and Invariants of Connections

We will continue on from the last section and consider the additional structure connections give us and how we can define invariants of a given connection. Again a good introduction can be found in [KN63] and [Mor01]. We will however describe all the details we need to define the moduli space of flat connections.

D.1 Covariant Derivative

The start of this section is slightly technical but gives us the basics we need to describe flat connections in terms of their connection 1-forms.

Definition: (Wedge Product)
Let M be a smooth manifold, G a Lie group with $Lie(G) = \mathfrak{g}$ the associated Lie algebra. Define the **wedge product** of Lie algebra valued forms for $k, l \in \mathbb{Z}_{\geqslant 0}$, $\alpha \in \Omega^k(M, \mathfrak{g})$ and $\beta \in \Omega^l(M, \mathfrak{g})$ and $X_1, ..., X_{k+l} \in \mathfrak{X}(M)$ we have

$$[\alpha \wedge \beta](X_1, ..., X_{k+l})$$
$$= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} sign(\sigma)[\alpha(X_{\sigma(1)}, ..., X_{\sigma(k)}), \beta(X_{\sigma(k+1)}, ..., X_{\sigma(k+l)})]$$

Notice that the wedge product is not completely antisymmetric however satisfies the following

$$[\alpha \wedge \beta] = (-1)^{kl+1}[\beta \wedge \alpha]$$

Definition: (Exterior Derivative)
Let M be a smooth manifold, G a Lie group with $Lie(G) = \mathfrak{g}$ the associated Lie algebra. We can analogously define the **exterior derivative** to satisfy the following

- We have $d_k : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g})$ a linear map
- For $f \in \Omega^0(M, \mathfrak{g}) = C^\infty(M, \mathfrak{g})$ we have and $d_0 f = df = Tf = f_*$ using the canonical association $T_v\mathfrak{g} \cong \mathfrak{g}$ so that $d_0 f : TM \to \mathfrak{g}$.

- $d_{k+1} \circ d_k = 0$

- For $\alpha \in \Omega^k(M, \mathfrak{g})$, $\beta \in \Omega^p(M, \mathfrak{g})$ we have $d_{k+p}[\alpha \wedge \beta] = [d_k(\alpha) \wedge \beta] + (-1)^k[\alpha \wedge d_p(\beta)]$.

Equivalently if $\omega \in \Omega^k(M, \mathfrak{g})$ is given locally for a chart (U, φ) by

$$\varphi^*(\omega|_U) = \sum_{0<i_1<...<i_k\leqslant n} f_{i_1,...,i_k}(x_1, ..., x_n)dx_{i_1} \wedge ... \wedge dx_{i_k}$$

then $d\omega \in \Omega^{k+1}(M, \mathfrak{g})$ is given by

$$\varphi^*(d\omega|_U) = \sum_{i_{k+1}=1}^{n} \left(\sum_{0<i_1<...<i_k\leqslant n} \frac{\partial f_{i_1,...,i_k}}{\partial x_{i_{k+1}}}(x_1, ..., x_n)dx_{i_{k+1}} \wedge dx_{i_1} \wedge ... \wedge dx_{i_k} \right)$$

For vector fields $X_1, ..., X_{k+1} \in \mathfrak{X}(M)$ we have the following where

$$(X_1, ..., \widehat{X_i}, ..., X_{k+1}) = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_{k+1})$$

denotes emission of the element X_i.

$$d\omega(X_1, ..., X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}X_i(\omega(X_1, ..., \widehat{X_i}, ..., X_{k+1}))$$

$$+ \sum_{1\leqslant i<j\leqslant k+1} (-1)^{i+j}\omega([X_i, X_j], X_1, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_{k+1})$$

From the various properties it can be shown that the exterior derivative is natural and equivariant in the following ways.

$$d \circ \varphi^* = \varphi^* \circ d$$

and

$$d \circ Ad_{g^{-1}} = Ad_{g^{-1}} \circ d$$

Definition: (Covariant Derivative)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let φ_{H_A} be the horizontal projection associated to A. Define the **covariant derivative** of A to be

$$d_A : \Omega^k(P, \mathfrak{g}) \to \Omega^{k+1}(P, \mathfrak{g}) \quad s.t$$

$$d_A(\beta)(X_1, ..., X_{k+1}) = d\beta\left(\varphi_{H_A}(X_1), ..., \varphi_{H_A}(X_{k+1})\right)$$

D.2 Curvature

Definition: (Curvature)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let ω_A be the connection 1-form associated to A. The **curvature** of A is defined as follows

$$F_A = d_A(\omega_A) \in \Omega^1(P, \mathfrak{g})$$

Lemma D.2.1. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle, $A \in \mathcal{A}_P$, ω_A be the connection 1-form associated to A and $F_A \in \Omega^2(P, \mathfrak{g})$ be the curvature associated to A. F_A is tensorial.*

Proof.

$$R_g^*(F_A)(X,Y) = R_g^*(d_A\omega_A)(X,Y) = d_A\omega_A(TR_g(X), TR_g(Y))$$

$$= d\omega_A(\varphi_{H_A} \circ TR_g(X), \varphi_{H_A} \circ TR_g(Y)) = d\omega_A(TR_g \circ \varphi_{H_A}(X), TR_g \circ \varphi_{H_A}(Y))$$
$$= (R_g^* \circ d)(\omega_A)(\varphi_{H_A}(X), \varphi_{H_A}(Y)) = (d \circ R_g^*)(\omega_A)(\varphi_{H_A}(X), \varphi_{H_A}(Y))$$
$$= d\left(Ad_{g^{-1}}\left(\omega_A(\varphi_{H_A}(X), \varphi_{H_A}(Y))\right)\right) = Ad_{g^{-1}}\left(d\omega_A(\varphi_{H_A}(X), \varphi_{H_A}(Y))\right)$$
$$= Ad_{g^{-1}}(d_A(\omega_A(X,Y))) = Ad_{g^{-1}}(F_A(X,Y))$$

Notice that for $X \in V \subseteq TP$ we have

$$F_A(X,Y) = d\omega_A(\varphi_{H_A}(X), \varphi_{H_A}(Y)) = d\omega_A(0, \varphi_{H_A}(Y)) = 0$$

By the antisymmetry of X and Y we see that F_A is tensorial 2-form. So following lemma C.2.1 there is some element of $\Omega^2(M, \mathfrak{g}_P)$ that represents the curvature F_A. □

Lemma D.2.2. *(Structure Equation of Maurer-Cartan)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle, $A \in \mathcal{A}_P$ and ω_A be the connection 1-form associated to A. We have the following equality for $X, Y \in \mathfrak{X}(P)$.

$$F_A(X,Y) = d\omega_A(X,Y) + [\omega_A(X), \omega_A(Y)]$$

Using the wedge product of Lie algebra valued forms we have

$$F_A = d\omega_A + [\omega_A \wedge \omega_A]$$

> **Remark:** This structure equation can be thought of as a generalisation of B.2.1 the relation satisfied by the exterior derivative of the Maurer-Cartan form which is the case when we have a principle bundle over a point. In this sense the one connection on the a principle bundle over a point has 0 curvature.

Proof. As $F_A(X,Y)$ and $d\omega_A(X,Y)+[\omega_A(X), \omega_A(Y)]$ are bilinear and antisymmetric with respect to X and Y we can reduce the proof to the following cases

- Case 1: X and Y are horizontal.
  That is $\varphi_{H_A}(X) = X$ and $\varphi_{H_A}(Y) = Y$ and so $\omega_A(X) = \omega_A(Y) = 0$ so

$$F_A(X,Y) = (d_A\omega_A)(X,Y) = d\omega_A(\varphi_{H_A}(X), \varphi_{H_A}(Y)) = d\omega_A(X,Y)$$

$$= d\omega_A(X,Y) + [0,0] = d\omega_A(X,Y) + [\omega_A(X), \omega_A(Y)]$$

- Case 2: X and Y are vertical.
  That is $\varphi_{V_A}(X) = X$ and $\varphi_{V_A}(Y) = Y$. Let $p \in P$ then $(\omega_A)_p(X) = T_p I_p(X_p)$ and $(\omega_A)_p(Y_p) = T_p I_p(Y_p)$. Let $u = T_p I_p(X_p)$ and $v = T_p I_p(Y_p)$. Let $u^*, v^* \in \mathfrak{X}(P)$ be the fundamental vector fields associated to u and v. Notice that $X_p = u_p$ and $Y_p = u_p$ and notice that we have $[u,v]^* = [u^*, v^*]$ and so $\omega_A([u,v]) = [\omega_A(u), \omega_A(v)]$. Also note that $\omega_A(u^*) = u$ and $\omega_A(v^*) = v$ and more importantly they are fixed and so $u_p(\omega_A(v)) = 0$ and $v_p(\omega_A(u)) = 0$. We then see that
  $$(F_A)_p(X_p, Y_p) = (d_A\omega_A)_p(X_p, Y_p) = (d\omega_A)_p(\varphi_{H_A}(X)_p, \varphi_{H_A}(Y)_p)$$
  $$= (d\omega_A)_p(0,0) = 0 = -\omega_A([u,v]_p) + [\omega_A(u)_p, \omega_A(v)_p]$$
  $$= u_p(\omega_A(v)) - v_p(\omega_A(u)) - (\omega_A)_p([u,v]_p) + [(\omega_A)_p(u_p), (\omega_A)_p(v_p)]$$
  $$= (d\omega_A)_p(u_p, v_p) + [(\omega_A)_p(u_p), (\omega_A)_p(v_p)] = (d\omega_A)_p(X_p, Y_p) + [(\omega_A)_p(X_p), (\omega_A)_p(Y_p)]$$
  So as $p \in P$ was arbitrary we can see that
  $$F_A(X,Y) = d\omega_A(X,Y) + [\omega_A(X), \omega_A(Y)]$$
- Case 3: X horizontal and Y vertical.
  That is $\varphi_{H_A}(X) = X$ and $\varphi_{V_A}(Y) = Y$ and so $(\omega_A)_p(X) = 0$ and $(\omega_A)_p(Y) = T_p I_p(Y_p)$. Let $p \in P$ and $v = T_p I_p(Y_p)$ and consider $v^* \in \mathfrak{X}(P)$ the fundamental vector field associated to v. Notice that $Y_p = v_p$ and $\omega_A(v^*) = v$ and more importantly it is fixed and so $X(\omega_A(v)) = 0$. We claim that $[X, v]$ is horizontal. The flow generated by v is given by $P \times \mathbb{R} \ni (p,t) \to p \cdot exp(tv)$ and so
  $$[X, v] = \lim_{t \to 0} \frac{TR_{exp(tT_pI_p(Y_p))}(X) - X}{t}$$
  By assumption X is horizontal and horizontal vectors are sent to horizontal vectors under the action of G so $TR_{exp(tT_pI_p(Y_p))}(X) - X$ is horizontal and so $[X, v] = \lim_{t \to 0} \frac{TR_{exp(tT_pI_p(Y_p))}(X) - X}{t}$ is horizontal. We now see that
  $$(F_A)_p(X_p, Y_p) = (d_A\omega_A)_p(X_p, Y_p) = (d\omega_A)_p(\varphi_{H_A}(X)_p, 0) = (d\omega_A)_p(X_p, 0) = 0$$
  $$= X_p(\omega_A(v^*)) - v^*_p(\omega_A(X)) - (\omega_A)_p([X, v^*]_p) + [0, (\omega_A)_p(Y_p)]$$
  $$= (d\omega_A)_p(X_p, v^*_p) + [0, (\omega_A)_p(Y_p)] = (d\omega_A)_p(X_p, Y_p) + [(\omega_A)_p(X_p), \omega_A(Y_p)]$$
  So as $p \in P$ was arbitrary we can see that
  $$F_A(X,Y) = d\omega_A(X,Y) + [\omega_A(X), \omega_A(Y)]$$

This completes the proof of the Structure Equation. The next lemma is proved by a similar method. ☐

Lemma D.2.3. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle, $A \in \mathcal{A}_P$ and ω_A be the connection 1-form associated to A. Let $\beta \in \Omega^1(P, \mathfrak{g})$ be a tensorial 1-form. Then we have the following equation for $X, Y \in \mathfrak{X}(P)$*
$$d_A\beta(X,Y) = d\beta(X,Y) + [\beta(X), \omega_A(Y)] + [\omega_A(X), \beta(Y)]$$
That is
$$d_A\beta = d\beta + 2[\beta \wedge \omega_A]$$

From the structure equation D.2.2 we can derive another expression for curvature.

Lemma D.2.4. *(Curvature)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Then

$$F_A(X, Y) = -\omega_A([\varphi_{H_A}(X), \varphi_{H_A}(Y)])$$

Proof.

$$\begin{aligned} F_A(X,Y) &= F_A(\varphi_{H_A}(X), \varphi_{H_A}(Y)) \\ &= d\omega_A(\varphi_{H_A}(X), \varphi_{H_A}(Y)) + [\omega_A(\varphi_{H_A}(X)), \omega_A(\varphi_{H_A}(Y))] \\ &= \varphi_{H_A}(X)(\omega_A(\varphi_{H_A}(Y)) - \varphi_{H_A}(Y)(\omega_A(\varphi_{H_A}(X)) - \omega_A([\varphi_{H_A}(X), \varphi_{H_A}(Y)]) + [0,0] \\ &= \varphi_{H_A}(X)(0) - \varphi_{H_A}(Y)(0) - \omega_A([\varphi_{H_A}(X), \varphi_{H_A}(Y)]) = -\omega_A([\varphi_{H_A}(X), \varphi_{H_A}(Y)]) \end{aligned}$$

□

> **Remark:** Notice that we can start to see something geometric that the curvature is capturing. The curvature is only 0 at a point if the horizontal tangent space associated to the connection is closed under the Lie bracket of vector fields at this point. This is sometimes called Frobenius integrability.

Lemma D.2.5. *(Bianchi's Identity)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Then

$$d_A F_A = 0$$

Proof. By the structure equation D.2.2 we have

$$\begin{aligned} d_A F_A(X_1, X_2, X_3) &= dF_A(\varphi_{H_A}(X_1), \varphi_{H_A}(X_2), \varphi_{H_A}(X_3)) \\ &= d(d\omega_A + [\omega_A \wedge \omega_A])(\varphi_{H_A}(X_1), \varphi_{H_A}(X_2), \varphi_{H_A}(X_3)) \\ &= d[\omega_A \wedge \omega_A](\varphi_{H_A}(X_1), \varphi_{H_A}(X_2), \varphi_{H_A}(X_3)) \end{aligned}$$

Now using the formula for the exterior derivative given here

$$d[\omega_A, \omega_A](\varphi_{H_A}(X_1), \varphi_{H_A}(X_2), \varphi_{H_A}(X_3))$$

$$\begin{aligned} &= \sum_{i=1}^{3} (-1)^{i+1} X_i([\omega_A \wedge \omega_A](\varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \varphi_{H_A}(X_3))) \\ &+ \sum_{1 \leqslant i < j \leqslant 3} (-1)^{i+j} [\omega_A \wedge \omega_A]([\varphi_{H_A}(X_i), \varphi_{H_A}(X_j)], \varphi_{H_A}(X_{j \neq k \neq i})) \end{aligned}$$

So noting that $\omega_A \circ \varphi_{H_A} = 0$ we see

$$d[\omega_A, \omega_A](\varphi_{H_A}(X_1), \varphi_{H_A}(X_2), \varphi_{H_A}(X_3)) = 0$$

□

D.3 Covariant Derivatives and Curvature

Lemma D.3.1. *(Covariant Derivative Chain Complex)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$.

$$d_A \circ d_A = 0 \quad \textit{if and only if} \quad F_A = 0$$

Proof. Let $F_A = 0$. We then see

$$(d_A \circ d_A\beta)(X_1, ..., X_{k+2})$$

$$= d(d_A\beta)(\varphi_{H_A}(X_1), ..., \varphi_{H_A}(X_{k+2})) = \sum_{i=1}^{k+2}(-1)^{i+1}\varphi_{H_A}(X_i)(d_A\beta(\varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \varphi_{H_A}(X_{k+2})))$$

$$+ \sum_{1 \leqslant i < j \leqslant k+2}(-1)^{i+j} d_A\beta([\varphi_{H_A}(X_i), \varphi_{H_A}(X_j)], \varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \widehat{\varphi_{H_A}(X_j)}, ..., \varphi_{H_A}(X_{k+2}))$$

$$= \sum_{i=1}^{k+2}(-1)^{i+1}\varphi_{H_A}(X_i)(d\beta(\varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \varphi_{H_A}(X_{k+2})))$$

$$+ \sum_{1 \leqslant i < j \leqslant k+2}(-1)^{i+j} d\beta(\varphi_{H_A}[\varphi_{H_A}(X_i), \varphi_{H_A}(X_j)], \varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \widehat{\varphi_{H_A}(X_j)}, ..., \varphi_{H_A}(X_{k+2}))$$

$$= d^2\beta(\varphi_{H_A}(X_1), ..., \varphi_{H_A}(X_{k+2}))$$

$$- \sum_{1 \leqslant i < j \leqslant k+2}(-1)^{i+j} d\beta(\varphi_{V_A}[\varphi_{H_A}(X_i), \varphi_{H_A}(X_j)], \varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \widehat{\varphi_{H_A}(X_j)}, ..., \varphi_{H_A}(X_{k+2}))$$

$$= \sum_{1 \leqslant i < j \leqslant k+2}(-1)^{i+j} d\beta(T_eL.(F_A(X_i, X_j)), \varphi_{H_A}(X_1), ..., \widehat{\varphi_{H_A}(X_i)}, ..., \widehat{\varphi_{H_A}(X_j)}, ..., \varphi_{H_A}(X_{k+2}))$$

Where the last equality uses the lemma D.2.4 and the fact $d^2 = 0$ and where we took $T_eL.(v)$ to be the vector field such that $(T_eL.(v))_p = T_eL_p(v)$. So if $F_A = 0$ then $d_A \circ d_A = 0$ by considering the last line in the equation for $(d_A \circ d_A\beta)(X_1, ..., X_{k+2})$.

If $d_A \circ d_A = 0$ then for $\beta \in \Omega^0(P, \mathfrak{g})$ and so $k = 0$ we have the following using the last line in the equation for $(d_A \circ d_A\beta)(X_1, ..., X_{k+2})$

$$(d_A \circ d_A\beta)(X, Y) = -d\beta(T_eL.(F_A(X, Y)))$$
$$= -T_eL.(F_A(X, Y))(\beta) = 0$$

This is true for all X, Y and β if and only if $F_A = 0$. □

Lemma D.3.2. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. If $\beta \in \Omega^k(P, \mathfrak{g})$ is pseudotensorial then*

- *$\beta \circ \varphi_{H_A}^{\otimes k}$ is tensorial*
- *$d\beta$ is pseudotensorial*
- *$d_A\beta$ is tensorial*

Proof. Suppose that $\beta \in \Omega^k(P, \mathfrak{g})$ is pseudotensorial. It follows from equivariance of φ_{H_A} that $R_g^*(\beta \circ \varphi_{H_A}^{\otimes k}) = \beta \circ \varphi_{H_A}^{\otimes k} \circ TR_g = \beta \circ TR_g \circ \varphi_{H_A}^{\otimes k} = R_g^*(\beta) \circ \varphi_{H_A}^{\otimes k} = Ad_{g^{-1}}(\beta \circ \varphi_{H_A}^{\otimes k})$. Similarly if $X \in \mathfrak{X}(V)$ and $X_i \in \mathfrak{X}(P)$ then $\beta \circ \varphi_{H_A}^{\otimes k}(X, X_2, ..., X_k) = \beta(\varphi_{H_A}(X), \varphi_{H_A}(X_2), ..., \varphi_{H_A}(X_k)) = \beta(0, \varphi_{H_A}(X_2), ..., \varphi_{H_A}(X_k)) = 0$.

We also have $R_g^*(d\beta) = d(R_g^*(\beta)) = d(Ad_{g^{-1}}(\beta)) = Ad_{g^{-1}}(d\beta)$.

The last dot point follows from the first two. □

Definition: (Affine Covariant Derivative)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Using the association in lemma C.2.1 between $\Omega^k(M, \mathfrak{g}_P)$ and the tensorial forms in $\Omega^k(P, \mathfrak{g})$ and the previous lemma D.3.2 we can then define the affine covariant derivative associated to the connection A. The **affine covariant derivative** is defined as follows

$$D_A : \Omega^k(M, \mathfrak{g}_P) \to \Omega^{k+1}(M, \mathfrak{g}_P) \quad s.t \quad D_A(\beta) \cong d_A\beta \in \Omega^k(P, \mathfrak{g})$$

Notice that from the lemma D.2.3 we have the following formula for the D_A

$$D_A(\beta) = d\beta + 2[\beta \wedge \omega_A]$$

Remark: It can be shown $D_A \circ D_A = 0$ if and only if F_A is zero as in lemma D.3.1. Using the association between tensorial forms on the bundle and adjoint valued forms on the base we will identify $d_A = D_A$.

Lemma D.3.3. *(Derivative of the Curvature Map)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Then for $\beta \in T_A\mathcal{A}_P \cong \Omega^1(M, \mathfrak{g}_P)$

$$T_AF(\beta) = d_A\beta$$

Proof. We will blur the line between $\Omega^1(M, \mathfrak{g}_P)$ and the tensorial 1-forms in $\Omega^1(P, \mathfrak{g})$. Notice from the structure equation D.2.2 that

$$F_{A+t\beta}(X, Y) = d\omega_{A+t\beta}(X, Y) + [\omega_{A+t\beta}(X), \omega_{A+t\beta}(Y)]$$

$$= d(\omega_A + t\beta)(X, Y) + [(\omega_A + t\beta)(X), (\omega_A + t\beta)(Y)]$$

$$= d\omega_A(X, Y) + [\omega_A(X) \wedge \omega_A(Y)] + t(d\beta(X, Y) + [\beta(X), \omega_A(Y)] + [\omega_A(X), \beta(Y)]) + t^2[\beta(X), \beta(Y)]$$

Now notice that from lemma D.2.3 we have $d\beta(X, Y) + [\beta(X), \omega_A(Y)] + [\omega_A(X), \beta(Y)] = d_A\beta(X, Y)$. So we see that

$$F_{A+t\beta} = F_A + d_At\beta + t^2[\beta \wedge \beta]$$

This means that $T_AF(\beta) = \lim_{t\to 0} \frac{F_{A+t\beta} - F_A}{t} = d_A\beta$. □

Appendix E

Holonomy

Holonomy and parallel transport could be described as the fundamental motivation mathematically to define connections. Given a path in the base we are interested in lifting the path into the bundle. There is no canonical way of lifting a given path in the base to a path in the bundle however given a point in a fibre and a connection there is a canonical way to lift the path to a horizontal path in the bundle. In the theory of affine connections on vector bundles this is used to compare tangent vectors at nearby points to take directional derivatives of vector fields via the affine covariant derivative.

E.1 Parallel Transport

Lemma E.1.1. *(Parallel Transport)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle. Let $A \in \mathcal{A}_P$ and let H_A be the horizontal sub-bundle of TP associated to A. Given a piecewise differentiable path $\gamma : [0,1] \to M$ and $p \in \pi^{-1}(\gamma(0))$ there exists a unique path $\widetilde{\gamma} : [0,1] \to P$ such that for $t \in I$ and $v \in T_t I$ we have $T_t\widetilde{\gamma}(v) \in H_{\widetilde{\gamma}(t)}$, $\widetilde{\gamma}(0) = p$ and $\pi \circ \widetilde{\gamma} = \gamma$.

Proof. We can reduce to this lemma to a local statement of P as we are only interested in the tangent vectors of the curve. Locally we can formulate the conditions $T_t\widetilde{\gamma}(v) \in H_{\widetilde{\gamma}(t)}$, $\widetilde{\gamma}(0) = p$ and $\pi \circ \widetilde{\gamma} = \gamma$ into differential equations with specified boundary values. Using results in the theory of differential equations locally there is a unique solution for the path $\widetilde{\gamma}$ and by compactness of $[0,1]$ we can extend this solution $\widetilde{\gamma}$ to the whole path. $\square$

Example: (Parallel Transport)

- Let $(\pi : M \times G \to M, G, \cdot)$ be a the trivial principle bundle and $A \in \mathcal{A}_{M\times G}$ be the trivial connection. Then for a piecewise differentiable path $\gamma : [0,1] \to M$ and $(\gamma(0), g) \in M \times G$ we have the following horizontal lift $\widetilde{\gamma} : [0,1] \to M \times G$ such that $\widetilde{\gamma}(t) = (\gamma(t), g)$.

- Let $\pi : U(1)\times U(1) \to U(1)$ such that $\pi(w, z) = w$ be the trivial principle $U(1)$-bundle over $U(1)$. Topologically this is the torus. Consider the connection A with horizontal subspace $H_A = \{(u, ru) : TU(1) \times TU(1)\}$ for $r \in \mathbb{R}$. Then for $\gamma : [0,1] \to U(1)$ such that $\gamma(t) = e^{2\pi it}$ we have $\widehat{\gamma}(t) = (e^{2\pi it}, e^{r2\pi it})$.

- Let $(\pi : M \times G \to M, G, \cdot)$ be a the trivial principle bundle and $A \in \mathcal{A}_{M\times G}$ be the connection with horizontal subspace given by $Tf(TM)$ for some section $f : M \to M \times G$. For $(\gamma(0), f(\gamma(0))g) \in M \times G$ we have the horizontal lift $\tilde{\gamma} : [0,1] \to M \times G$ such that $\tilde{\gamma}(t) = R_g \circ f \circ \gamma(t)$. This example covers the last with the trivial section. This is important as locally flat connections are described by the trivial connection.

Remark: Loops in M generally do not lift to loops in P. This then leads us to the following definition.

Definition: (Holonomy Group)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let $PWC^\infty([0,1], \gamma(0) = \gamma(1), M, x)$ be the set of piecewise smooth paths based at x. For $\gamma \in PWC^\infty([0,1], \gamma(0) = \gamma(1), M, x)$ and $p \in \pi^{-1}(x)$ let $\tilde{\gamma} \in PWC^\infty([0,1], P, p)$ be the horizontal lift of γ based at p.

$$Hol_{P,p}(A) = \{g \in G : \text{for } \gamma \in PWC^\infty([0,1], \gamma(0) = \gamma(1), M, x) \text{ we have } \tilde{\gamma}(1) = p \cdot g\}$$

Note that this is a subspace as we can compose paths and take inverse paths to find that the set $Hol_{P,p}(A)$ is indeed a subspace. We call $Hol_{P,p}(A)$ the **holonomy group of A at p.**

Lemma E.1.2. *Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$.*

$$Hol_{P,p\cdot g}(A) = Ad_{g^{-1}}\left(Hol_{P,p}(A)\right) = g^{-1} Hol_{P,p}(A) g$$

Proof. Let consider $h \in Hol_{P,p\cdot g}(A)$. So there exists $\gamma \in PWC^\infty([0,1], \gamma(0) = \gamma(1), M, \pi(x))$ and $p \in \pi^{-1}(x)$ with $\tilde{\gamma} \in PWC^\infty([0,1], P, p)$ the horizontal lift of γ based at p with $\tilde{\gamma}(1) = p\cdot h$. Then $R_g \circ \tilde{\gamma}$ is horizontal piecewise continuous curve with $R_g \circ \tilde{\gamma}(0) = p\cdot g$ and $\pi \circ R_g \circ \tilde{\gamma} = \pi \circ \tilde{\gamma} = \gamma$ and so is the horizontal lift of γ based at $p\cdot g$. We have $R_g \circ \tilde{\gamma}(1) = p \cdot h \cdot g = (p \cdot g) \cdot (g^{-1} h g)$. So $Hol_{P,p\cdot g}(A) \leqslant Ad_{g^{-1}}\left(Hol_{P,p}(A)\right)$. A similar argument in reverse shows that $Ad_g\left(Hol_{P,p\cdot g}(A)\right) \geqslant Hol_{P,p}(A)$ and so $Hol_{P,p\cdot g}(A) \geqslant Ad_{g^{-1}}\left(Hol_{P,p}(A)\right)$. ∎

Definition: (Infinitesimal Holonomy Group)
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_P$. Let $\mathcal{U}_p = \{U \subseteq M : \text{for open } U \text{ and } \pi(p) \in U\}$. We have a directed system by inclusion for $U \subseteq V$ and $U, V \in \mathcal{U}_p$ we have $Hol_{\pi^{-1}(U),p}(A|_U) \leqslant Hol_{\pi^{-1}(V),p}(A|_V)$. The colimit of this directed system defines the **infinitesimal holonomy group** at p as follows

$$Hol_p(A) = \bigcap_{U \in \mathcal{U}_p} Hol_{\pi^{-1}(U),p}(A|_U)$$

Remark: The infinitesimal holonomy group $Hol_p(A)$ is trivial if and only if the connection A is flat at p. That is $(F_A)_p = 0$.

Example: (Holonomy)

- Let $(\pi : M \times G \to M, G, \cdot)$ be the trivial bundle and $A \in \mathcal{A}_P$ the trivial connection. Then $Hol_{P,p}(A) = \{e\} \leqslant G$ and similarly $Hol_p(A) = \{e\}$.
- Let $\pi : U(1) \times U(1) \to U(1)$ such that $\pi(w, z) = w$ be the trivial principle $U(1)$-bundle over $U(1)$. Consider the connection A with horizontal subspace $H_A = \{(u, ru) : TU(1) \times TU(1)\}$ for $r \in \mathbb{R}$. For $r = a + \frac{b}{c} \in \mathbb{Q}$ with $a, b, c \in \mathbb{Z}$ with $|b| < |c|$ we have $Hol_{P,p}(A) = \mathbb{Z}/c\mathbb{Z}$. For $r \in \mathbb{R}-\mathbb{Q}$ we have $Hol_{P,p}(A) = \mathbb{Z}$. However in all cases for $r \in \mathbb{R}$ we have $Hol_p(A) = \{e\}$.

E.2 Holonomy and Representations of the Fundamental Group

Lemma E.2.1. *(Flat Connections and Homotopy)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_{P,flat}$. Let γ_0 and γ_1 be piecewise smooth paths in M with a homotopy $H : [0,1] \times [0,1] \to M$ such that $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$ with $\widetilde{\gamma_0}$ and $\widetilde{\gamma_1}$ being the horizontal lifts of γ_0 and γ_1. Then the homotopy lifts to a homotopy $\widetilde{H} : [0,1] \times [0,1] \to P$ such that $\widetilde{H}(t, 0) = \widetilde{\gamma_0}(t)$ and $\widetilde{H}(t, 1) = \widetilde{\gamma_1}(t)$

Proof. By definition flat connections are locally the given by the trivial bundle with trivial connection. Homotopies horizontally lift on the trivial bundle with trivial connection. By compactness of $[0, 1]$ we can therefore horizontally lift homotopies into a bundle with a flat connection. □

Corollary E.2.2. *(Representations of the Fundamental Group)*
Let $(\pi : P \to M, G, \cdot)$ be a principle bundle and $A \in \mathcal{A}_{P,flat}$. Consider $x \in M$ and $p \in \pi^{-1}(x)$. For the connection A we can canonically define an element $\rho_A \in Hom(\pi_1(M, x), G)/G$ where G acts on $Hom(\pi_1(M, x), G)$ such that $(\rho \cdot g)(x) = g^{-1}\rho(x)g$.

> **Remark:** Notice that with the extra information of $p \in \pi^{-1}(x)$ we have a representation ρ_p called the holonomy representation of the connection A of the fundamental group based at x with respect to p.

Proof. For $[\gamma] \in \pi_1(M, x)$ from lemma E.2.1 we can define an element $\rho_p([\gamma]) \in Hol_{P,p}(A)$. Notice that if we had instead chosen $p \cdot g \in \pi^{-1}(x)$ from lemma E.1.2 we get $\rho_{p \cdot g}([\gamma]) = g^{-1}\rho_p([\gamma])g$. □

The question is then raised as to what extent this conjugacy class of representation determines the flat connection and if all conjugacy classes of representations can be achieved by holonomy representation.

Appendix F

Symplectic Geometry

Symplectic Geometry is a modern take on classical Hamiltonian mechanics. Symplectic manifolds are more flexible than Riemannian Manifolds and have no local invariants. However there are some important properties that separate them from smooth manifolds. These rigidity properties are exemplified in Gromov non-squeezing theorem , the Gromov-Eliashberg's Rigidity Theorem and Arnold's Conjecture. See [MS95] for more details on the foundational aspects of symplectic geometry another good reference is [dS02]. This is also covered in [Aud04] which is most relevant for our purposes.

F.1 Classical Mechanics

As we mentioned in the introduction to Appendix F symplectic geometry is a modern take on classical Hamiltonian mechanics. Hamiltonian mechanics is a method of modelling classical (i.e non-quantum or "big") mechanical systems. For the classical examples arising in the study of mechanics, such as free body motion and motion where the force is specified by some potential, Hamiltonian mechanics can be used to derive Newtons laws and vice versa. In fact its not much of a shift from the ideas of Newton and is more of a rephrasing. The ideas of Hamilton are more general than we will present but these are the basic ideas.

Example: (Hamilton's Equations for n Particles in 1-Dimension)
We want to consider n particles in 1-dimensional space. The position and momentum of these particles specifies $2n$ real parameters. We assume this completely determines the system (i.e there are no other degrees of freedom).

Let $M = \mathbb{R}^{2n} = \{(q_1, p_1, ..., q_n, p_n) : q_1, p_1, ..., q_n, p_n \in \mathbb{R}\}$ be the phase space of our system where we view q_i as the position of the i-th particle and p_i as the momentum of the i-th particle. When modelling a particular 1-dimensional system of n particles we try to guess a Hamiltonian function $H : M \to \mathbb{R}$ that is supposed to represent the energy of the system.

Given a point $x \in M$ representing the state of the classical system at time 0 we want to consider a function $\gamma = (\gamma_{q_1}, \gamma_{p_1}, ..., \gamma_{q_n}, \gamma_{p_n}) : \mathbb{R} \to M$ such that $\gamma(0) = x$

and γ satisfies Hamilton's equations which are the following

$$\left.\frac{d\gamma_{q_i}}{dt}\right|_t = \left.\frac{\partial H}{\partial p_i}\right|_{\gamma(t)} \quad \text{and} \quad \left.\frac{d\gamma_{p_i}}{dt}\right|_t = -\left.\frac{\partial H}{\partial q_i}\right|_{\gamma(t)}$$

Given the Hamiltonian or energy H of the system if γ satisfies Hamilton's equations then $\gamma(t)$ represents the state of the system at time t notice that for $t = 0$ we have $\gamma(0) = x$ the initial state of the system. Again we stress that Hamilton's equations are equivalent to Newton's equations in most contexts of interest.

We are interested in what are called observable quantities associated to our system. In particular we are interested in $f \in C^\infty(M)$ which will represent any quantity related to our system we could potentially measure in an experiment. Consider the function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$. We are interested in how the observable quantity f changes in time (i.e the derivative of $f \circ \gamma$). We have

$$\begin{aligned}\left.\frac{df \circ \gamma}{dt}\right|_t &= \sum_{i=1}^n \left.\frac{df}{dq_i}\right|_{\gamma(t)} \left.\frac{d\gamma_{q_i}}{dt}\right|_t + \left.\frac{df}{dp_i}\right|_{\gamma(t)} \left.\frac{d\gamma_{p_i}}{dt}\right|_t \\ &= \sum_{i=1}^n \left.\frac{df}{dq_i}\right|_{\gamma(t)} \left.\frac{\partial H}{\partial p_i}\right|_{\gamma(t)} - \left.\frac{df}{dp_i}\right|_{\gamma(t)} \left.\frac{\partial H}{\partial q_i}\right|_{\gamma(t)}\end{aligned}$$

Using this as motivation we define the Poisson bracket $\{-,-\} : C^\infty(M) \otimes_{\mathbb{R}} C^\infty(M) \to C^\infty(M)$ such that for $f, g \in C^\infty(M)$ and $x \in M$ we have

$$\{f,g\}(x) = \sum_{i=1}^n \left.\frac{df}{dq_i}\right|_x \left.\frac{\partial g}{\partial p_i}\right|_x - \left.\frac{df}{dp_i}\right|_x \left.\frac{\partial g}{\partial q_i}\right|_x$$

It can be shown that $\{-,-\}$ defines the structure of a real Lie algebra on $C^\infty(M)$. Notice that $C^\infty(M)$ is also an algebra with point wise multiplication. It can also be seen that $\{-,-\}$ satisfies the Leibniz rule. That is for $f, g, h \in C^\infty(M)$

$$\{fg, h\} = f\{g,h\} + g\{f,h\}$$

So using the Poisson bracket we can see that the an observable $f \in C^\infty(M)$ changes in time as follows.

$$\left.\frac{df \circ \gamma}{dt}\right|_t = \{f, H\}(\gamma(t))$$

This is the general scheme of classical mechanics. All of the quantities of interest are then calculated by solving the various differential equations. For example conservation of energy is simply proven as follows (note that we define H as the energy so the use of the word proven is slightly circular)

$$\left.\frac{dH}{dt}\right|_t = \{H, H\}(\gamma(t)) = 0$$

To generalise this we want to consider all possible phase spaces not just Euclidean space. This means we also have to generalise Hamilton's equations and the Poisson bracket. Notice that if we choose $H(q_1, p_1, ..., q_n, p_n) = q_i$ or $H(q_1, p_1, ..., q_n, p_n) = p_i$ then Hamilton's equations become

$$\left.\frac{d\gamma_{q_i}}{dt}\right|_t = 0 \quad \text{and} \quad \left.\frac{d\gamma_{p_i}}{dt}\right|_t = -1$$

or

$$\left.\frac{d\gamma_{q_i}}{dt}\right|_t = 1 \quad \text{and} \quad \left.\frac{d\gamma_{p_i}}{dt}\right|_t = 0$$

An interesting object is then the two form

$$\omega = dq_1 \wedge dp_1 + ... + dq_n \wedge dp_n \in \Omega^2(M)$$

This is a non-degenerate closed two form. We can rephrase the above equations as follows. As ω is non-degenerate for $f \in C^\infty(\mathbb{R}^{2n})$ there exists a unique vector field $X_f \in \mathfrak{X}(\mathbb{R}^{2n})$ such that $\omega(X_f, v) = df(v)$. This is called the Hamiltonian vector field associated to f because we have

$$\left.\frac{d\gamma}{dt}\right|_t = (X_H)_{\gamma(t)}$$

$$\left.\frac{df \circ \gamma}{dt}\right|_t = \{f, H\}(\gamma(t)) = \omega_{\gamma(t)}(X_f, X_H) = T_{\gamma(t)}H(X_f) = -T_{\gamma(t)}f(X_H)$$

This is only a small part of the theory of classical mechanics. For more on the physics see the classic text [GPS50].

F.2 Basic Definitions and Results

Definition: (Symplectic Manifold)
Let M be a smooth manifold. Let $\omega \in \Omega^2(M)$. If ω is closed and non-degenerate

- $d\omega = 0$
- For $X_x \in T_xM - \{0\}$ there exists $Y_x \in T_xM$ such that $\omega_x(X_x, Y_x) \neq 0$.

then we say (M, ω) is a **symplectic manifold**.

Remark: ω_x is an antisymmetric non-degenerate bilinear form on T_xM. By induction it can be shown that this implies $dim(T_xM)$ is even. This shows that symplectic manifolds must have even dimension.

Example: (Symplectic Manifold)

- The fundamental example is given by $M = \mathbb{R}^{2n} = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}\}$ and $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$.
- A well studied example is given by $M = \Sigma_g$ a compact orientable surface of genus g and ω an area form.

- Let N be a smooth n-manifold and take $M = T^*N$ where we let $\pi : M = T^*N \to N$ be the projection associated to the tangent bundle. We define $\lambda \in \Omega^1(M)$ such that $\lambda_x(X_x) = x(T_x\pi(X_x))$ where we recall that $x \in T^*N$ and $T_x\pi(X_x) \in TN$. Then take $\omega = d\lambda$.

  Definition: (Symplectomorphism)
  Let (M, ω) and (N, ν) be a symplectic manifolds. Let $f : M \to N$ be a smooth map. We say f is a **symplectomorphism** if it preserves the symplectic form, that is
  $$f^*\nu = \omega$$
  Let $Aut(M, \omega)$ be the group of symplectomorphisms from (M, ω) to (M, ω).

The following theorems describe the local invariants of a symplectic manifold. They will show that in fact the only local invariant is the dimension of the manifold in stark contrast to Riemannian geometry where one encounters curvature.

Theorem F.2.1. *(Weinstein's Theorem using Moser's Trick)*
Let M be a smooth manifold. Let $V \subseteq M$ be a closed sub-manifold and let $\omega_0, \omega_1 \in \Omega^2(M)$ be two symplectic forms on M such that for $v \in V$ we have $(\omega_0)_v = (\omega_1)_v$. Then there exists an open set $U \subseteq M$ such that $V \subseteq U$ and a map

$$\psi : U \to M \quad s.t \quad \psi|_V = id_V \text{ and for } v \in V \quad (\psi^*\omega_1)_v = (\omega_0)_v$$

Corollary F.2.2. *(Darboux Theorem)*
Let (M, ω) be a symplectic $2n$-manifold and let $x \in M$. Then there exists and open set $U \subseteq M$ such that $x \in U$ with a chart from an open set $\Omega \subseteq \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}\}$ given by $\varphi : \Omega \to U$ such that

$$\varphi^*(\omega) = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$$

Proof. Consider the following path in $\Omega^2(M)$,

$$[0, 1] \ni t \mapsto \omega_t = \omega_0 + t(\omega_1 - \omega_0)$$

Notice that ω_t is closed for all t as $d\omega_t = d(\omega_0 + t(\omega_1 - \omega_0)) = d\omega_0 + t(d\omega_1 - d\omega_0) = 0$. Also notice that for $v \in V$ we have $(\omega_t)_v = (\omega_0)_v = (\omega_1)_v$. This means that ω_t is non-degenerate on V.

Now for $v \in V$ we have $(\omega_t)_v$ is non-degenerate so there exists vector fields $X, Y \in \mathfrak{X}(M)$ such that $(\omega_t)_v(X_v, Y_v) \neq 0$. Now as $M \ni x \mapsto (\omega_t)_x(X_x, Y_x)$ is smooth there exists an open set $U_v \subseteq M$ such that $v \in U_v$ and for $x \in U_v$ we have $(\omega_t)_x(X_x, Y_x) \neq 0$. In other words an open neighbourhood of v where ω_t is non-degenerate.

Taking the union over $v \in V$ we have an open set $U' \subseteq M$ such that $V \subseteq U'$ and $\omega_t|_{U_t}$ is non-degenerate. So in fact $\omega_t|_{U_t}$ is a symplectic form on U_t. Now as $[0, 1]$ is compact we can find a neighbourhood U' such that $\omega_t|_{U'}$ is symplectic for all $t \in [0, 1]$.

Now we can choose another open set $U'' \subseteq U'$ such that $V \subseteq U''$ and where V is in fact a deformation retract of U''. To see this simply take a small ball around each $v \in V$ contained in the open set U_v. Let $r : U'' \times I \to U''$ be a deformation retraction.

Let $\iota_Y(\omega_t)(X) = \omega_t(Y, X)$ and take $(X_t)_x = T_t r_x \left(\frac{\partial}{\partial t}\right)$. We take the following homotopy operator $Q : \Omega^2(U'') \to \Omega^1(U'')$ such that $Q(\omega)(X) = \int_0^1 r_t^*(\iota_{X_t}\omega)(X)dt$. We then have

$$d(Q(\omega_1 - \omega_0)) = Q(d(\omega_1 - \omega_0)) + d(Q(\omega_1 - \omega_0))$$

$$= \int_0^1 r_t^*(\iota_{X_t} d(\omega_1 - \omega_0))dt + \int_0^1 dr_t^*(\iota_{X_t}(\omega_1 - \omega_0))dt$$

$$= \int_0^1 r_t^*(\iota_{X_t} d(\omega_1 - \omega_0) + d(\iota_{X_t}(\omega_1 - \omega_0)))dt = \int_0^1 r_t^*(\mathcal{L}_{X_t}(\omega_1 - \omega_0))dt$$

$$= \int_0^1 \frac{d}{dt} r_t^*(\omega_1 - \omega_0)dt = r_1^*(\omega_1 - \omega_0) - r_0^*(\omega_1 - \omega_0) = -(\omega_1 - \omega_0)$$

where we used the Cartan magic formula, the definition of the Lie derivative and the fundamental theorem of calculus. Let $\beta = Q(\omega_1 - \omega_0)$ and notice that for $v \in V$ we have $\beta_v = 0$.

Now as ω_t is non-degenerate we can define a vector field $Y_t \in \mathfrak{X}(U'')$ for $t \in [0, 1]$ such that $\omega_t(Y_t, X) = \beta(X)$. Notice that non-degeneracy implies that for $v \in V$ we must have $(Y_t)_v = 0$. Let φ_t be the flow of the vector field Y_t and notice that for φ_t must therefore fix V. Therefore there exists a neighbourhood of V say U such that $\varphi_t(U) \subseteq U$. Therefore we see that

$$\frac{d}{dt}\left(\varphi_t^*\omega_t\right) = \varphi_t^*\left(\frac{d\omega_t}{dt} + \mathcal{L}_{Y_t}(\omega_t)\right) = \varphi_t^*\left(\frac{d\omega_0 + t(\omega_1 - \omega_0)}{dt} + \iota_{Y_t} d\omega_t + d(\iota_{Y_t}\omega_t)\right)$$

$$= \varphi_t^*\left((\omega_1 - \omega_0) + d\beta\right) = \varphi_t^*\left((\omega_1 - \omega_0) + (\omega_0 - \omega_1)\right) = 0$$

So $\varphi_t^*\omega_t$ is constant with respect to t. This means that $\omega_0 = \varphi_0^*\omega_0 = \varphi_t^*\omega_t = \varphi_1^*\omega_1$. So let $\psi = \varphi_1$ to get the statement of the theorem.

To prove the corollary we use an exponential map associated to some Riemannian metric to associate the standard symplectic form on the tangent space at a point $x \in M$ induced by ω_x to a neighbourhood around x and compare this to ω. We then apply the theorem where $V = x$. □

We have now described our phase space where classical mechanics takes place that is a symplectic manifold (M, ω). This may now have interesting global structures related to the topology of M and the symplectic form ω but locally this phase space is the same as the classical example described in section F.1. Now we formalise the other information relevant when studying classical mechanics. We need a Hamiltonian or Energy function and then we need to determine the time evolution of the classical system given a state at time zero. This will correspond to the flow through phase space.

Definition: (Hamiltonian Flow and Hamiltonian Vector Fields)
Let (M, ω) be a symplectic manifold and let $H \in C^\infty(M)$. Consider the vector field $X_H \in \mathfrak{X}(M)$ such that for $Y \in \mathfrak{X}(M)$ we have

$$\omega(X_H, Y) = dH(Y)$$

This exists and is unique as ω is non-degenerate. Now let $\varphi_H : M \times \mathbb{R} \to M$ be the flow of X_H. We say that φ_H is the **Hamiltonian flow** associated to the Hamiltonian H and X_H is the **Hamiltonian vector field** associated to the Hamiltonian H.

Definition: (Hamiltonian System)
Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$. We say that (M, ω, H) is a **Hamiltonian system**.

Example: The classic examples arising in physics are given by

- Newtonian gravity in 1-dimension where $M = \mathbb{R}^2$ and $\omega = dx \wedge dy$ with $H(x, y) = \frac{1}{2m}y^2 + mgx$ where $m \in \mathbb{R}$ is the mass and $g \in \mathbb{R}$ is the acceleration due to gravity. Here $\frac{1}{2m}y^2$ corresponds to the kinetic energy and mgx the gravitational potential energy.
- the harmonic oscillator where $M = \mathbb{R}^{2n}$ and $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ with $H(x_1, y_1, ..., x_n, y_n) = \frac{1}{2m}y_1^2 + ... + \frac{1}{2m}y_1^2 + \frac{1}{2}kx_1^2 + ... + \frac{1}{2}kx_n^2$. Here $\frac{1}{2m}y^2$ corresponds to the kinetic energy and $\frac{1}{2}kx^2$ the spring potential energy.
- These examples fall into the following class of examples. Let $M = \mathbb{R}^{2n}$, $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ and $H(x_1, y_1, ..., x_n, y_n) = \frac{1}{2m}y_1^2 + ... + \frac{1}{2m}y_1^2 + V(x_1, ..., x_n)$ for some potential $V \in C^\infty(\mathbb{R}^n)$.

Definition: (Poisson Bracket)
Let (M, ω) be a symplectic manifold. Define the **Poisson bracket** associated to (M, ω) to be the following

$$\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \quad s.t \quad \{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f)$$

where $X_f, X_g \in \mathfrak{X}(M)$ are the vector fields defined such that $\omega(X_f, Y) = df(Y)$ and $\omega(X_g, Y) = dg(Y)$.

Remark: Hamilton's equations then become $\frac{df \circ \varphi_H}{dt} = \{f, H\} \circ \varphi_H$ where φ_H is the Hamiltonian flow of H which is a consequence of the chain rule as $\{f, H\}(\varphi_H(x, t)) = T_{\varphi_H(x,t)}f(X_H)$ but $d\varphi(\frac{d}{dt}) = \frac{d\varphi_H}{dt} = X_H$ and so $\frac{df \circ \varphi_H}{dt} = df \circ d\varphi(\frac{d}{dt}) = df(X_H) = \{f, H\}$. What this shows is that really we are interested in the Hamiltonian flow of H as this corresponds to the time evolution of the system when given $x \in M$ as initial conditions.

We have the following abstract definition.

Definition: (Poisson Algebra)
Let A be an algebra over k. If $[\ ,\ ] : A \times A \to A$ is a Lie bracket on A that satisfies the Leibniz rule

$$[ab, c] = a[b, c] + [a, c]b$$

Then we say $(A, [\ ,\ ])$ is a **Poisson algebra**.

Definition: (Poisson Manifold)
Let M be a smooth manifold. Consider the algebra over $\mathbb{R}$ given by $C^\infty(M)$. If $(C^\infty(M), \{\ ,\ \})$ is a Poisson algebra then we say that $(M, \{\ ,\ \})$ is a **Poisson manifold**.

Remark: Every symplectic manifold is a Poisson manifold with the Poisson bracket defined via Hamiltonian flow above.

Definition: (Casimir Elements)
Let $(M, \{\ ,\ \})$ be a Poisson manifold. We say $f \in C^\infty(M)$ is **Casimir** if $\{f, \cdot\} = 0$ or in other words the set of Casimir functions is given by the center of the underlying Lie algebra.

Remark: Poisson manifolds are foliated by symplectic sub-manifolds with symplectic leaves having fixed values for all the Casimir functions.

F.3 Symplectic Group Actions, the Moment Map and Symplectic Quotients

Definition: (Symplectic Group Actions)
Let (M, ω) be a symplectic manifold. Let G be a group that acts on M smoothly. We say that G **acts symplectically** if for $x \in M$ and $g \in G$ we have $R_g^*(\omega) = \omega$ where $R_g : M \to M$ such that $R_g(x) = x \cdot g$. As always with group actions we can think of the action as a homomorphism into the group of symplectomorphisms.

Definition: (Hamiltonian S^1 and $\mathbb{R}$ Actions)
Let (M, ω) be a symplectic manifold. Let $G = U(1)$ or $G = \mathbb{R}$ act on M symplectically. Consider $L_x : G \to M$ such that $L_x(t) = x \cdot t$ and the vector field X such that $X_x = T_e L_x(v)$ for $v \in T_e G$. We say that the **G-action is Hamiltonian** if there exists a function H_v such that $X_{H_v} = X$.

We can extend this definition to Cartesian products of $U(1)$ and $\mathbb{R}$. However this definition is generalised to arbitrary Lie group as follows.

Definition: (Hamiltonian G Actions and the Moment Map)
Let (M, ω) be a symplectic manifold and G be a Lie group acting symplectically on (M, ω). We say that the **G-action is Hamiltonian** if there exists a map called the **moment map** $\mu : M \to \mathfrak{g}^*$ such that

- For $v \in \mathfrak{g} = T_eG$ and $v^* = X$ such that $X_x = T_eL_x(v)$ and for $\mu^v \in C^\infty(M)$ given by $\mu^v(x) = \mu(x)(v)$ that for $Y \in \mathfrak{X}(M)$ we have $d\mu^v(Y) = \omega(X, Y)$.

Remark: The condition says that for each 1-parameter subgroup if we restrict the action the S^1 or $\mathbb{R}$-action is Hamiltonian in the previous sense.

Remark: For $f \in \mathfrak{g}^*$ then $\mu + f$ is another moment map.

Remark: If for $g \in G$ we have $\mu \circ R_g = Ad_g^* \circ \mu$ in other words μ is equivariant then we say that μ is an equivariant moment map.

Remark: We have two cases that are of great interest.

- G is semi-simple where any symplectic action is Hamiltonian and equivariant moment maps are unique.

- G is Abelian where symplectic actions may not be Hamiltonian the coadjoint action is trivial so there is a $\mathfrak{g}^*$ worth of equivariant moment maps.

For the details on these facts see lecture 4 in [dS02].

Example: (Hamiltonian Actions)

- Take $M = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}\} = \mathbb{R}^{2n} = \mathbb{C}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n\}$ with symplectic form given by $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$. We have a symplectic action of $U(1)$ on M such that $(z_1, ..., z_n) \cdot e^{2\pi i\theta} = (z_1 e^{2\pi i\theta}, ..., z_n e^{2\pi i\theta})$. This can be generalised to a $G = U(1)^n$ action such that for $(e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}) \in U(1)^n$ we have $(z_1, ..., z_n) \cdot (e^{2\pi i\theta_1}, ..., e^{2\pi i\theta_n}) = (z_1 e^{2\pi i\theta_1}, ..., z_n e^{2\pi i\theta_n})$. A moment map is given by $\mu(x_1, y_1, ..., x_n, y_n) = \frac{1}{2}(x_1^2 + y_1^2, ..., x_n^2 + y_n^2) \in \mathbb{R}^n \cong \mathfrak{g}^*$
- $M = T^*\mathbb{R}^3 = \{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6\}$ and $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3$. Let $SO(3)$ act on $\mathbb{R}^3$ in the standard way. Then we can lift the action to an action on M. This is Hamiltonian with moment map $\mu(x_1, x_2, x_3, y_1, y_2, y_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$. Notice that μ is the angular momentum about 0.
- Take $M = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}\} = \mathbb{R}^{2n} = \mathbb{C}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n\}$. Let $U(n)$ act in the standard way on $\mathbb{C}^n$. This is Hamiltonian with moment map

$$\mu(z_1, ..., z_n) = \frac{i}{2} \begin{bmatrix} z_1\overline{z_1} & \dots & z_1\overline{z_n} \\ \vdots & & \vdots \\ z_n\overline{z_1} & \dots & z_n\overline{z_n} \end{bmatrix} \in \mathfrak{u}(n) \cong \mathfrak{u}(n)^*$$

Definition: (Hamiltonian G-Space)
Let (M, ω) be a symplectic manifold and G a Lie group with a Hamiltonian action with moment map μ. We call (M, ω, G, μ) a **Hamiltonian G-space**.

We wish to define a quotient in the category of symplectic manifolds. On the outset there is no canonical way to do this. Purely on a dimensional argument if we have a free action of an odd dimensional Lie group then the quotient manifold would be of odd dimension and wouldn't have a chance of being symplectic. The moment map now enables us to define a symplectic quotient almost canonically for a Hamiltonian G-space.

Theorem F.3.1. *(Symplectic Quotient)*
Let (M, ω, G, μ) be a Hamiltonian G-space with compact G with μ equivariant. If the G-action on $\mu^{-1}(0)$ is free then $\mu^{-1}(0)/G$ is a smooth manifold and there exists $\omega_{red} \in \Omega^2(\mu^{-1}(0)/G)$ such that $(\mu^{-1}(0)/G, \omega_{red})$ is a symplectic manifold. Moreover if $\iota : \mu^{-1}(0) \hookrightarrow M$ is the inclusion and $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is the quotient map then $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principle bundle and $\iota^*(\omega) = \pi^*(\omega_{red})$.

Proof. The proof breaks into four different parts.

$\mu^{-1}(0)$ is a manifold of dimension $dim(M) - dim(G)$:
For $x \in \mu^{-1}(0)$ let $\mathfrak{stab}(x) = \{v \in \mathfrak{g} : T_eL_x(v) = 0\} = ker(T_xL_x)$ be the Lie algebra of $Stab(x)$. Notice that by the definition of the moment map for $X_x \in T_xM$ and $v \in \mathfrak{g}$ we have $\omega_x(T_eL_x(v), X_x) = T_x\mu^v(X_x)$ and so $ker(T_x\mu) = \{X_x \in T_xM : T_x\mu = 0 \in \mathfrak{g}^*\}$ using the canonical identification of $T_{\mu(x)}\mathfrak{g}^* = \mathfrak{g}^*$. Using this identification if $X_x \in ker(T_x\mu)$ then for all $v \in \mathfrak{g}$ we have $\omega_x(T_eL_x(v), X_x) = T_x\mu^v(X_x) = 0$. So $ker(T_x\mu) = \{X_x \in T_xM : \text{if } v \in \mathfrak{g} \text{ then } \omega_x(T_eL_x(v), X_x)\}$.

Now notice that $im(T_x\mu) \subseteq Ann(\mathfrak{stab}(x))$ as for $v \in \mathfrak{stab}(x)$ we have $T_eL_x(v) = 0$ and so $T_x\mu^v(X_x) = \omega_x(T_eL_x(v), X_x) = \omega_x(0, X_x) = 0$ where we are again using the canonical identification of $T_{\mu(x)}\mathfrak{g}^* = \mathfrak{g}^*$.

Now counting dimensions $dimker(T_x\mu) + dimim(T_x\mu) = dim(T_xM)$. Notice that $dimker(T_x\mu) = dim(T_xM) - dimim(T_eL_x)$ and that $dimker(T_eL_x) + dimim(T_eL_x) = dim(\mathfrak{g})$. This means that $dimim(T_x\mu) = dim(\mathfrak{g}) - dimker(T_eL_x) = dim(\mathfrak{g}) - dim(\mathfrak{stab}(x))$.

Now we can see that $dim(Ann(\mathfrak{stab}(x))) = dim(\mathfrak{g}) - dim(\mathfrak{stab}(x))$ by taking a basis for $\mathfrak{stab}(x)$ extending that to a basis for $\mathfrak{g}$ and then taking the dual basis of that basis. So by the dimension count $dimim(T_x\mu) = dim(Ann(\mathfrak{stab}(x)))$ and so $im(T_x\mu) = Ann(\mathfrak{stab}(x))$.

Let G act freely on $\mu^{-1}(0)$. We see that $Stab(x) = 0$ and therefore that $im(T_x\mu) = Ann(\mathfrak{stab}(x)) = Ann(\{0\}) = \mathfrak{g}^*$. Then $T_x\mu$ is surjective for all $x \in \mu^{-1}(0)$. Therefore 0 is a regular value of μ. So $\mu^{-1}(0) \subseteq M$ is a sub-manifold of dimension $dim(M) - dim(G)$.

$\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ is a principle G-bundle:
Using the slice theorem we can show a manifold with a free group action of a compact Lie group G is the same as the information of a principle bundle. See chapter 1 of [Aud04] or proposition 5.4 in [dS02] for more details.

Definition of ω_{red}:
We have $T_x(x \cdot G) \subseteq T_x(\mu^{-1}(0))$ as $x \cdot G \subseteq \mu^{-1}(0)$. Notice that $T_x\mu^{-1}(0)/T_x(x \cdot G) \cong T_{[x]}(\mu^{-1}(0)/G)$. We can then define $\omega_{red} \in \Omega^2(\mu^{-1}(0)/G)$ such that for $[X]_{[x]}, [Y]_{[x]} \in T_x\mu^{-1}(0)/T_x(x \cdot G) = T_{[x]}(\mu^{-1}(0)/G)$ with $x \in \mu^{-1}(0)$ such that $\pi(x) = [x]$ and $X_x, Y_x \in T_x\mu^{-1}(0)$ such that $T_x\pi(X_x) = [X]_{[x]}$ and $T_x\pi(Y_x) = [Y]_{[x]}$ we have $(\omega_{red})_{[x]}([X]_{[x]}, [Y]_{[x]}) = \omega_x(X_x, Y_x)$. Notice that x and X_x, Y_x exist as π is surjective.

ω_{red} is well defined:
We have $T_x(\mu^{-1}(0)) = ker(T_x\mu) = \{X_x \in T_xM : \text{if } v \in \mathfrak{g} \text{ then } 0 = \omega_x(T_eL_x(v), X_x) = T_x\mu^v(X_x)\} = \{X_x \in T_xM : \text{if } Y_x \in T_x(x \cdot G) \text{ then } 0 = \omega_x(Y_x, X_x)\}$. This shows for $W_x, Z_x \in T_x(x \cdot G)$ we have $\omega_x(X_x, Y_x) = (\omega_{red})_{[x]}([X]_{[x]}, [Y]_{[x]}) = (\omega_{red})_{[x]}([X + W]_{[x]}, [Y + Z]_{[x]}) = \omega_x(X_x + W_x, Y_x + Z_x) = \omega_x(X_x, Y_x) + \omega_x(X_x, Z_x) + \omega_x(W_x, Y_x) + \omega_x(W_x, Z_x) = \omega_x(X_x, Y_x) + 0 + 0 + 0 = \omega_x(X_x, Y_x)$.

ω_{red} is symplectic:
ω_{red} is non-degenerate as if $(\omega_{red})_{[x]}([X]_{[x]}, [Y]_{[x]}) = 0$ for all $[Y]_x$ then $\omega_x(X_x, Y_x) = 0$ for all $Y_x \in T_x\mu^{-1}(0)$ and so $X \in T_x(x \cdot G)$ and so $[X]_{[x]} = 0$. We can see that from the construction that $\iota^*(\omega) = \pi^*(\omega_{red})$ and as π^* is injective we can see that $0 = \iota^*(d\omega) = d \circ \iota^*(\omega) = d \circ \pi^*(\omega_{red}) = \pi^*(d\omega_{red})$ means $d\omega_{red} = 0$. $\square$

> **Remark:** The symplectic quotient is not unique and depends on the moment map. One can also define the quotient by taking $\mu^{-1}(f)$ for a regular value of μ such that $Ad^*_{g^{-1}}(f) = f$. However the different symplectic quotients are often related.

Classical Mechanics and the Symplectic Quotient (Reduction): Suppose that we have Hamiltonian system and Hamiltonian G-space (M, ω, H, G, μ) with H a G-invariant function or in other words $H(x \cdot g) = H(x)$. In physics the action of G represents a symmetry of the mechanical system given by (M, ω, H) and μ is representing some conserved quantity of Hamiltonian flow that comes from the symmetry of G.

Notice that for translations and rotations of the standard symplectic space that the moment map gives the momentum and angular momentum respectfully. These symmetries represent a redundancy in the description of the system. Given an initial state (point in M) which determines the value of the moment map what the symplectic quotient does is remove this redundancy in the description of the system.

Symbolically given (M, ω, H, G, μ) with H equivariant and $x \in M$ with $\mu(x)$ a regular value of μ where G acts freely on $\mu^{-1}(\mu(x))$. Here x is the initial point of our system. Let $\gamma_{H,x}$ be the flow of X_H such that $\gamma_{H,x}(0) = x$ and notice that $\gamma_H : \mathbb{R} \to \mu^{-1}(\mu(x))$ and that $\gamma_H \cdot g = \gamma_{H,x \cdot g}$. This means that $(\mu^{-1}(\mu(x))/G, \omega_{red}, H)$ with $[x] \in \mu^{-1}(\mu(x))/G$ has the same dynamics as (M, ω, H) with dimension reduced by $2dim(G)$ which makes computations often much simpler.

In the language of symplectic geometry Noether's principle states that there is a natural one to one correspondence between the symmetries associated to one-parameter subgroups of G and G-invariant functions which is achieved by the Hamiltonian flow of G-invariant functions. See [GPS50] for more on the physics perspective.

Example: (Symplectic Quotients)

- Let $M = \{(x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}\} = \mathbb{R}^{2n}$ and $\omega = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ with $\mathbb{R}^m$-action for $m < n$ given by $(x_1, y_1, ..., x_n, y_n) \cdot (a_1, ..., a_m) = (x_1 + a_1, y_1, ..., x_m + a_m, y_m, x_{m+1}, y_{m+1}, ..., x_n, y_n)$ with moment map

$$\mu(x_1, y_1, ..., x_n, y_n) = (y_1, ..., y_m) \in \mathbb{R}^m = (\mathbb{R}^m)^*$$

Consider

$$\mu^{-1}(0) = \{(x_1, 0, ..., x_m, 0, x_{m+1}, y_{m+1}, ..., x_n, y_n) \in \mathbb{R}^{2n}\} \cong \mathbb{R}^{2n-m}$$

and

$$\begin{aligned}\mu^{-1}(0)/\mathbb{R}^m &= \{([0], 0, ..., [0], 0, x_{m+1}, y_{m+1}, ..., x_n, y_n)\} \\ &\cong \{(x_{m+1}, y_{m+1}, ..., x_n, y_n) \in \mathbb{R}^{2n-2m}\} = \mathbb{R}^{2n-2m}\end{aligned}$$

With the identification of $\mu^{-1}(0)/\mathbb{R}^m \cong \{(x_{m+1}, y_{m+1}, ..., x_n, y_n) \in \mathbb{R}^{2n-2m}\}$ we have $\omega_{red} = dx_{m+1} \wedge dy_{m+1} + ... + dx_n \wedge dy_n$.

- Let $M = \mathbb{C}^\times \times \mathbb{C}^{n-1}$ and ω the standard symplectic form with $G = U(1)$ acting on M such that $(z_1, ..., z_n) \cdot e^{2\pi i\theta} = (z_1 e^{2\pi i\theta}, ..., z_n)$. Consider coordinates $r_1 \in (0, \infty)$ and $\theta_1 \in U(1)$ with $z_1 = re^{2\pi i\theta_1}$. Then $2\omega = 2r_1 dr_1 \wedge d\theta_1 - dz_2 \wedge d\overline{z_2} - ... - dz_n \wedge d\overline{z_n}$ and for the moment map μ, $2\pi i \in \mathfrak{u}(1) = 2\pi i\mathbb{R}$ and $x = (r_1, \theta_1, z_2, ..., z_n) \in M$ we have moment map $T_x\mu^{2\pi i}(X_x) = \omega_x(r_1\frac{\partial}{\partial\theta_1}, X_x) = -\frac{1}{2}r_1 X_x^{r_1}$. So $T_x\mu^{2\pi i} = \frac{1}{2}r_1 dr_1$ and we see $\mu = \frac{1}{2}r_1^2 + c = \frac{1}{2}(x_1^2 + y_1^2) + c$ for $c \in \mathbb{R}_{<0}$. We then have $\mu^{-1}(0)/G = \mathbb{C}^{n-1}$ and $\omega_{red} = dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n$.

- Let $M = \mathbb{C}^{k\times n}$ and ω the standard symplectic form with $G = U(k)$ acting on the by left matrix multiplication. This has moment map $\mu(A) = \frac{i}{2}AA^* + \frac{Id}{2i}$ where we use the canonical identification $\mathfrak{u}(k) = \mathfrak{u}(k)^*$ (via the killing form). Then we get a symplectic form ω_{red} on $\mu^{-1}(0)/G = Gr_{\mathbb{C}}(k, n)$ the grassmannian manifold of k-planes in $\mathbb{C}^n$.

  Definition: (G-Actions on Poisson Manifolds)
  Let $(M, \{\ ,\ \})$ be a Poisson manifold and G a smooth Lie group. If G acts on M such that for $g \in G$ and $R_G : M \to M$ such that $R_g(x) = x \cdot g$ then $\{f \circ R_g, g \circ R_g\} = \{f, g\}$ then we say that G is a Poisson action.

  Definition: (Poisson Manifold Quotient)
  Let $(M, \{\ ,\ \})$ be a Poisson manifold and G a smooth Lie group that has a Poisson action on $(M, \{\ ,\ \})$. If M/G induces a manifold then we can define the quotient Poisson manifold such that for $f, g \in C^\infty(M/G)$ there exists $\widetilde{f}, \widetilde{g} \in C^\infty(M)$ such that $\widetilde{f}(x) = f([x])$ and $\widetilde{g}(x) = g([x])$. Then $\{f, g\}_G = \{\widetilde{f}, \widetilde{g}\}$ defines a Poisson bracket such that $(M/G, \{\ ,\ \}_G)$ is a Poisson manifold.

Remark: If the quotient of a symplectic manifold by a symplectic action is a manifold then there is a natural Poisson structure on the quotient manifold and if the action is Hamiltonian then the Poisson structure is in fact foliated by the pre-image of the coadjoint orbits in $\mathfrak{g}^*$. See chapter III of [Aud04] for more on these results.

F.4 Toric Symplectic Manifolds and the Duistermaat-Heckman Theorem

There are many beautiful results concerning toric varieties. We will list a few important facts that will be needed in the calculation of the volume of the moduli space of flat connections. We will also state the Duistermaat-Heckman formula. See [dS01] for the toric symplectic manifolds and [dS02] also. For the Duistermaat-Heckman theorem see [AB84] and [Aud04].

Theorem F.4.1. *(The Atiyah-Guillemin-Sternberg Convexity Theorem)*
Let $T = U(1)^k$ be a torus with $Lie(T) = \mathfrak{t}$ and (M, ω, T, μ) be a Hamiltonian T-space with μ equivariant. Then we have the following

- *$\mu^{-1}(v)$ is connected for all $v \in \mathfrak{t}^*$.*
- *$\mu(M)$ is the convex hull of the images of the fixed points of the action.*

Remark: We will construct a symplectic structure on the moduli space of $SU(2)$-flat connections in section 2.1.2 and exhibit a Hamiltonian torus action on the moduli space in section 2.2.4. We will see that the images under the moment map will be constructed out of the moduli space $\mathcal{R}_{SU(2),0,3}$ which we have seen is a convex polytope in lemma 1.2.9.

Theorem F.4.2. *(Duistermaat-Heckman Theorem)*
Let $T = U(1)^k$ be a torus with $Lie(T) = \mathfrak{t}$ and (M, ω, T, μ) be a Hamiltonian T-space with μ equivariant. Then $\mu_* \left(\frac{\omega^n}{n!}\right)$ is a piecewise polynomial multiple of the Lebesgue measure on $\mathfrak{t}^*$ with highest degree given by $\frac{dim(M)}{2} - k$.

Remark: Notice that the support of the measure will be a polytope from the previous theorem.

Remark: This result becomes in some sense trivial when one introduces equivariant cohomology. A fantastic account of this is given in [AB84].

Theorem F.4.3. *(Half Dimensional Torus Actions and Duistermaat-Heckman)*
Let $T = U(1)^k$ be a torus with $Lie(T) = \mathfrak{t}$ and (M, ω, T, μ) be a Hamiltonian T-space with μ equivariant. If $k = \frac{dim(M)}{2}$ then we have

$$Vol_{\frac{\omega^n}{n!}}(M) = Vol_{Euc}(\mu(M))$$

Where the Euclidean volume on $\mathfrak{t}$ is defined to assign volume 1 the fundamental domain of the period lattice $\Lambda \subseteq \mathfrak{t}$ such that $T = \mathfrak{t}/\Lambda$.

Remark: This last theorem will be used to calculate the volumes of our moduli space.

Appendix G

Quantum Field Theories

G.1 Partition Functions in QFT

We will sketch some of the ideas related to path integration in Quantum Field Theory and how this leads to the formal definition of TQFTs. We will follow the outline of the theory described in the introduction of [Koh02].

Definition: (σ-Models)
Let X be a Riemannian manifold. The σ-model over X associated to a compact oriented $(d+1)$-dimensional Riemannian manifold M is defined to be the classical physical system

- with states given by fields $\phi : M \to X$ with $\phi \in C^\infty(M, X)$

- with Action given by the Dirichlet functional $S_M : C^\infty(M, X) \to \mathbb{R}$ such that $S_M(\phi) = \int_X \|d\phi\|^2 dM$ for dM the volume form on M.

Remark: Fields that minimise the Action are called harmonic. These fields describe the dynamics of the classical system.

Proposition G.1.1. *(Action's and Locality)*
The Dirichlet functional $S_M : C^\infty(M, X) \to \mathbb{R}$ such that $S_M(\phi) = \int_X \|d\phi\|^2 dM$ satisfies the following properties.

- *For $f : M \to N$ an isometry for the induced map $f^* : C^\infty(N, X) \to C^\infty(M, X)$ we have $S_M(f^*(\phi)) = S_N(\phi)$.*

- *For M^* denoting M with the reversed orientation we have $S_{M^*}(\phi) = -S_M(\phi)$*

- *If $M = M_1 \cup_\Sigma M_2$ for some d-dimensional manifold Σ (note this could be empty) then $S_M(\phi) = S_{M_1}(\phi|_{M_1}) + S_{M_2}(\phi|_{M_2})$.*

Remark: We can see that gluing together σ-models satisfies nice properties with respect to the Action.

In physics it is often helpful when describing a quantum mechanical system to look for a classical analogue of the system and quantise. On standard way to do this is via Feynman path integrals. These path integrals calculate correlation functions. Let $J \in C^\infty(M, X)$. Then we symbolically represent the path integral as follows

$$Z(J) = \int_{C^\infty(M,X)} \exp\left(\frac{i}{\hbar} S_M(\phi) + \frac{i}{\hbar}\int_M \|J\|^2\|\phi\|^2\right) D\phi$$

Correlation functions are gotten by differentiating with respect to J. In defining invariants of the space time or in particular M we are interested in $Z(0) = Z$ given by

$$Z = \int_{C^\infty(M,X)} \exp\left(\frac{i}{\hbar} S_M(\phi)\right) D\phi$$

Notice that both of these quantities are ill defined as we have not specified a measure on $C^\infty(M, X)$. This is often hard to do and these theories are often not mathematically well defined. To use physical intuition to build mathematically rigorous theories we abstract these definitions to a well defined set of axioms. We then check if rigorously given constructions satisfy similar structures as these partition functions. The first key to the mathematical construction is by considering manifolds with boundary.

For $v \in C^\infty(\partial M, X)$ we can define let $C^\infty_v(M, X) = \{f \in C^\infty(M, X) : f|_{\partial M} = v\}$ and then define

$$Z(v) = \int_{C^\infty_v(M,X)} \exp\left(\frac{i}{\hbar} S_M(\phi)\right) D\phi$$

We want to consider $Z : C^\infty(\partial M, X) \to \mathbb{C}$. We want to define a vector space $Z(\partial M)$ such that we can identify $Z \in Z(\partial M)$. As previously mentioned we want to abstract the properties that such a vector space $Z(\partial M)$ would satisfy. The properties of the action listed in proposition G.1.1 will give us the following properties of $Z(\partial M)$ and $Z = Z(M)$.

Proposition G.1.2. *(Locality of the Partition Function)*

- *If $\partial M = \partial M_1 \sqcup \partial M_2$ we have and $v|_{\partial M_1} = v_1$ and $v|_{\partial M_2} = v_2$ we have*

$$Z(\partial M)(v) = \int_{C^\infty_v(M,X)} \exp\left(\frac{i}{\hbar} S_M(\phi)\right) D\phi$$

$$= \int_{C^\infty_{v_1}(M_1,X)} \int_{C^\infty_{v_2}(M_1,X)} \exp\left(\frac{i}{\hbar} S_{M_1}(\phi_1) + \frac{i}{\hbar} S_{M_2}(\phi_2)\right) D\phi_1 D\phi_2$$

$$= \left(\int_{C^\infty_{v_1}(M_1,X)} \exp\left(\frac{i}{\hbar} S_{M_1}(\phi_1)\right) D\phi_1\right) \left(\int_{C^\infty_{v_2}(M_1,X)} \exp\left(\frac{i}{\hbar} S_{M_2}(\phi_2)\right) D\phi_1 D\phi_2\right)$$

$$= Z(\partial M_1)(v_1) Z(\partial M_2)(v_2)$$

So we want to take $Z(\partial M) = Z(\partial M_1) \otimes Z(\partial M_2)$

- *For $M = M_1 \cup_\Sigma M_2$ we have*

$$Z(M)$$
$$= \int_{C^\infty(M,X)} \exp\left(\frac{i}{\hbar} S_M(\phi)\right) D\phi$$
$$= \int_{v \in C^\infty(\Sigma,X)} \int_{C_v^\infty(M_2,X)} \int_{C_v^\infty(M_1,X)} \exp\left(\frac{i}{\hbar} S_{M_1}(\phi_1) + \frac{i}{\hbar} S_{M_2}(\phi_2)\right) D\phi_1 D\phi_2 Dv$$
$$= \int_{v \in C^\infty(\Sigma,X)} \left(\int_{C_v^\infty(M_1,X)} \exp\left(\frac{i}{\hbar} S_{M_1}(\phi_1)\right) D\phi_1\right) \left(\int_{C_v^\infty(M_2,X)} \exp\left(+\frac{i}{\hbar} S_{M_2}(\phi_2)\right) D\phi_2\right) Dv$$
$$= \int_{v \in C^\infty(\Sigma,X)} Z(M_1)(v) Z(M_2)(v)$$

We therefore want to define $Z(\Sigma^*) = Z(\Sigma)^*$ and $Z(M) = \langle Z(M_1), Z(M_2)\rangle$.

Remark: One can play with the different properties such an integral would satisfy. These integrals can be useful heuristic tools when it comes to understanding different aspects of the topology of the space-time or underlying manifold. In section G.2 we will however develop an axiomatic approach to QFT. The particular types of QFTs we will study will in the sense of the partition functions above come from special kinds of Actions. For TQFT the Actions of the theory will be invariant under diffeomorphism. For CFT on the other hand the Action will be invariant under conformal transformations.

G.2 Topological Field Theories

We give a slightly refined version of Atiyah's functorial definition of a Topological Quantum Field Theory. This was based on Segal's definition of Conformal Field Theory via modular functors. We will firstly need the following definition.

Definition: (Cobordism Category)
We define the category of $d+1$ dimensional cobordisms denoted $\underline{Cob_{d+1}}$ to be the category with

- $ob(\underline{Cob_{d+1}}) = \{$isomorphism classes of oriented d-manifolds$\}$
- For $\Sigma_1, \Sigma_2 \in ob(\underline{Cob_{d+1}})$ we have

$$Hom_{\underline{Cob_{d+1}}}(\Sigma_1, \Sigma_2) = \Big\{\text{isomorphism classes of oriented } (d+1)\text{-manifolds } M$$
$$\text{with } \partial M \cong \Sigma_1^* \sqcup \Sigma_2 \text{ and } f_1 : \Sigma_1^* \hookrightarrow \partial M \text{ and } f_2 : \Sigma_2 \hookrightarrow \partial M\Big\} = \{(M, f_1, f_2)\}$$

where Σ_1^* denotes Σ_1 with the reversed orientation.

- The composition is given by glueing. That is

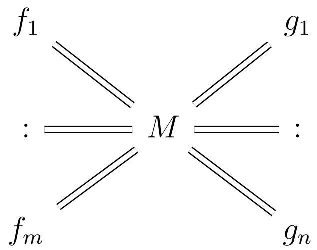
$$(M, f_1 : \Sigma_1^* \hookrightarrow \partial M, f_2 : \Sigma_2 \hookrightarrow \partial M) \circ (N, g_2 : \Sigma_2^* \hookrightarrow \partial M, g_3 : \Sigma_3 \hookrightarrow \partial M)$$
$$= (M \cup_{f_2,g_2} N, f_1 : \Sigma_1^* \hookrightarrow \partial M \cup_{f_2,g_2} N, f_2 : \Sigma_3 \hookrightarrow \partial M \cup_{f_2,g_2} N)$$

where $M \cup_{f_2,g_2} N = M \sqcup N/\sim$ with $x \sim y$ if there exists $z \in \Sigma_2$ such that $f_2(z) = x$ and $g_2(z) = y$ with canonical smooth structure and as f_1, g_3 map to $M \sqcup N$ with image away form the gluing they descend to maps into $M \cup_{f_2,g_2} N$.

Remark: The inclusions f_1 and f_2 are key to defining the category. They allow us to glue together manifolds along common boundaries as if we didn't specify the embeddings we would not canonically be able to glue together manifolds.

Remark: With the operation of disjoint union and the empty d-manifold makes $\underline{Cob}_{d+1}$ a monoidal category. In fact we can find cobordisms that form twist morphisms which makes $\underline{Cob}_{d+1}$ a symmetric monoidal category.

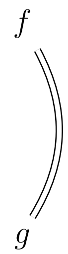
Definition: (Pictorial Representations)
For $i = 1, ..., m$ let $\Sigma_{1,i}$ be a closed smooth connected d-manifold and for $i = 1, ..., n$ let $\Sigma_{2,i}$ be a closed smooth connected d-manifold. Let M be a smooth $(d+1)$-manifold with $\partial M \cong \Sigma_{1,1} \sqcup \Sigma_{1,m} \sqcup \Sigma_{2,1} \sqcup ... \sqcup \Sigma_{2,n}$. $(M, f_1 \sqcup ... \sqcup f_m, g_1 \sqcup ... \sqcup g_n)$ and let $f_i : \Sigma^*_{1,i} \hookrightarrow \partial M$ and $g_i : \Sigma_{2,i} \hookrightarrow \partial M$. We can represent $(M, f_1 \sqcup ... \sqcup f_m, g_1 \sqcup ... \sqcup g_n)$ pictorially as follows.



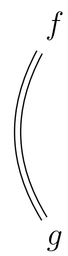
We can use these pictures to keep track of the gluing in $\underline{Cob_{d+1}}$.

We will also use the special pictorial representations

$(\Sigma \times [0,1], f \sqcup g, \varnothing) \quad = $



$((\Sigma \times [0,1])^*, \varnothing, f \sqcup g) \quad = $



$((\Sigma \times [0,1])^*, f, g) \quad = \quad f =\!= g$

Definition: (Topological Quantum Field Theory)(TQFT)
Let R be a commutative ring and let $\underline{RMod}$ be the category of finitely

generated R-modules. A $(d+1)$-dimensional TQFT over R is a functor $Z : \underline{Cob_{d+1}} \to \underline{RMod}$ satisfying the following axioms

- For $\Sigma \in ob(\underline{Cob_{d+1}})$ we have $Z(\Sigma^*) = Z(\Sigma)^*$. This says reversing orientations corresponds to taking algebraic duals in $\underline{RMod}$.

- For $\Sigma_1, \Sigma_2 \in ob(\underline{Cob_{d+1}})$ we have $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes_R Z(\Sigma_2)$. This says that the monoidal structure in $\underline{Cob_{d+1}}$ corresponds to the monoidal structure in $\underline{RMod}$.

- For $(M, f_1, f_2) \in Hom_{\underline{Cob_{d+1}}}(\Sigma_1, \Sigma_2)$ and $(N, g_1, g_2) \in Hom_{\underline{Cob_{d+1}}}(\Sigma_2, \Sigma_3)$ we have $Z((M, f_1, f_2) \circ (N, g_1, g_2)) = Z(M, f_1, f_2) \circ Z(N, g_1, g_2)$. This says that Z preserves the composition.

- For $\varnothing \in ob(\underline{Cob_{d+1}})$ we have $Z(\varnothing) = R$. This says that the identity in $\underline{Cob_{d+1}}$ is sent to the identity in $\underline{RMod}$.

- For $\Sigma \in ob(\underline{Cob_{d+1}})$ and $(\Sigma \times [0,1], Id_{\Sigma*}, Id_\Sigma) \in Hom_{\underline{Cob_{d+1}}}(\Sigma, \Sigma)$ we have $Z(\Sigma \times [0,1], Id_{\Sigma*}, Id_\Sigma) = id_{Z(\Sigma)} \in Hom_{\underline{Cob_{d+1}}}(Z(\Sigma), Z(\Sigma))$. Noting that $(\Sigma \times [0,1], Id_{\Sigma*}, Id_\Sigma) = id_\Sigma \in Hom_{\underline{Cob_{d+1}}}(\Sigma, \Sigma)$ this says that Z sends identity morphisms to identity morphism.

Remark: The axioms state that Z is a symmetric monoidal functor from the symmetric monoidal category $\underline{Cob_{d+1}}$ to the symmetric monoidal category $\underline{RMod}$.

Remark: The axioms allow us to determine the TQFT on disconnected cobordisms by considering the connected components. Therefore we are only really interested in the connected cobordims.

We now prove an immediate consequence of the axioms which will prove that we must take finitely generated R-modules in our definition.

Lemma G.2.1. *(Trace Formula)*
Let $Z : \underline{Cob_{d+1}} \to \underline{RMod}$ be a TQFT. Suppose that for all $\Sigma \in ob(\underline{Cob_{d+1}})$ that $Z(\Sigma)$ is a free R-module. Consider $\Sigma \in ob(\underline{Cob_{d+1}})$ and $f : \Sigma \to \Sigma$. Now consider $(\Sigma \times [0,1], id_\Sigma^*, f) \in Hom_{\underline{Cob_{d+1}}}(\Sigma, \Sigma)$. We then get $Z(\Sigma \times I, id_\Sigma^*, f) = Z_f : Z(\Sigma) \to Z(\Sigma)$. Let $\Sigma_f = \Sigma \times [0,1]/\sim$ where $(f(x), 0) \sim (x, 1)$. Then we have

$$Tr(Z_f) = Z(\Sigma_f)$$

Corollary G.2.2. *(Dimension Formula)*
We then see that for $f = id_\Sigma$ we get

$$Z(\Sigma \times S_1) = Z(\Sigma_{id_\Sigma}) = Tr(Z_{id_\Sigma}) = Tr(id_{Z(\Sigma)}) = \dim(Z(\Sigma))$$

Proof. Let $v_1, ..., v_n$ be free generators for $Z(\Sigma)$ and let $f_1, ..., f_n$ be the dual basis such that $f_i(v_j) = \delta_{ij}$. Then we have for $(\Sigma \times [0,1], \varnothing, id_\Sigma \sqcup f)$ some element

$$Z(\Sigma \times [0,1], \varnothing, id_\Sigma \sqcup f) = \sum_{i,j=1}^{n} a_{ij} v_i \otimes v_j \in Z(\Sigma) \otimes Z(\Sigma)$$

Now for $(\Sigma \times [0,1], id^*_{\Sigma \sqcup \Sigma}, \varnothing)$ we have some element

$$Z(\Sigma \times [0,1], id^*_{\Sigma \sqcup \Sigma}, \varnothing) = \sum_{k,l=1}^{n} b_{ij} f_i \otimes f_j \in Z(\Sigma)^* \otimes Z(\Sigma)^*$$

By the following gluing

$$\begin{array}{ccc} f & & \\ \Big(& & \\ id_\Sigma & - & id_{\Sigma *} \\ & & \Big) \\ & & id_{\Sigma *} \end{array}$$

we must have

$$Z_f = \sum_{i,j,k,l=1}^{n} a_{ij} b_{kl} f_k(v_j) v_i \otimes f_l = \sum_{i,j,l=1}^{n} a_{ij} b_{kl} v_i \otimes f_l$$

So we see that

$$Z_f(v_p) = \sum_{i,j,l=1}^{n} a_{ij} b_{jl} f_l(v_p) v_i = \sum_{i,j=1}^{n} a_{ij} b_{jp} v_i$$

So representing Z_f as a matrix we have $(Z_f)_{ip} = \sum_{j=1}^{n} a_{ij} b_{jp}$. Notice that if we take the following gluing

$$\begin{array}{ccc} f & - & id_{\Sigma *} \\ \Big(& & \Big) \\ id_\Sigma & - & id_{\Sigma *} \end{array}$$

that we get

$$Z(\Sigma_f) = \sum_{i,j,k,l=1}^{n} a_{ij} b_{kl} f_k(v_i) f_l(v_j) = \sum_{i,j=1}^{n} a_{ij} b_{ji} = \sum_{i=1}^{n} (Z_f)_{ii} = Tr(Z_f)$$

□

Remark: (Infinitely Generated TQFTs)
This lemma shows that if we had a TQFT with some $\Sigma \in ob(\underline{Cob_{d+1}})$ such that $Z(\Sigma)$ was free then it must be finitely generated. Otherwise in the calculation above we would eventually need to take the trace of an infinite dimensional matrix. Which for the identity matrix is not well defined.

This shows that defining TQFT's over finitely generated R-modules was not just a choice but a necessity. The key axiom that requires this is that $Z(\Sigma \times [0,1], Id_{\Sigma *}, Id_\Sigma) = id_{Z(\Sigma)}$.

The issue with infinite dimensionality holds more generally when one studies the categorical properties of $\underline{Cob_{d+1}}$.

G.3 1+1 Dimensional Topological Field Theory

We will briefly describe the correspondence between Frobenius algebra's and $(1+1)$-dimensional TQFTs. There is a good introduction in [Law96] and slightly more detailed exposition in [Koc04].

> **Remark:** Notice that $ob(\underline{Cob_{1+1}}) = \{\varnothing\} \cup \{(S^1)^{\sqcup n}\}_{n \in \mathbb{Z}_{>0}}$. Therefore a (1+1)-dimensional TQFT has only one interesting R module $Z(S^1)$.

> **Remark:** There is only one way two glue two circles together so we can suppress the notation in $Hom_{\underline{Cob_{1+1}}}$ as this won't depend on the choice of maps. The only information stored in the maps is the in and out boundaries.

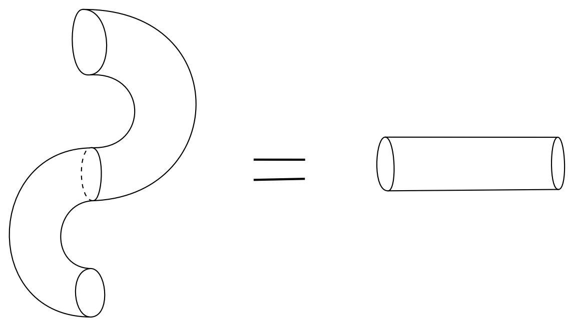
Lemma G.3.1. *(Cylinder and a Bilinear Form)*
Let Z be a $(1+1)$-dimensional TQFT over R. Then

$$Z(\Sigma_{0,2}, id_{(S^1)^*} \times id_{(S^1)^*}, \varnothing) : Z(S^1) \otimes Z(S^1) \to R$$

is non-degenerate.

Proof. Follows from gluing rules described in the following picture and the fact that

$$Z(\Sigma_{0,2}, id_{(S^1)^*}, id_{S^1}) = id_{Z(S^1)}$$



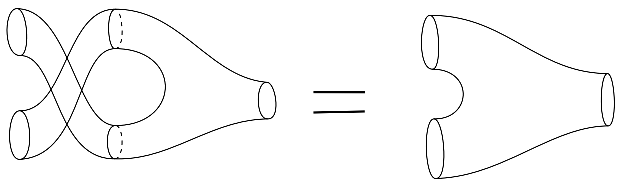
□

Lemma G.3.2. *(Pair of Pants and the Product)*
Let Z be a $(1+1)$-dimensional TQFT over R. Then

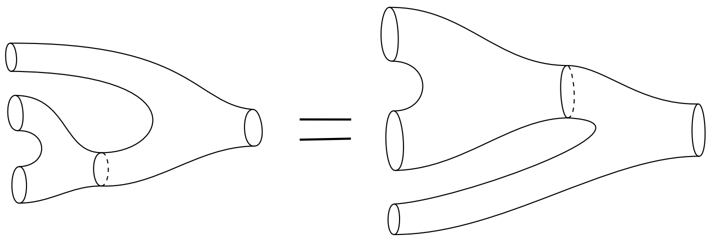
$$Z(\Sigma_{0,3}, id_{(S^1)^*} \times id_{(S^1)^*}, id_{S^1}) : Z(S^1) \otimes Z(S^1) \to Z(S^1)$$

defines a commutative product.

Proof. Commutativity follows from the following picture.



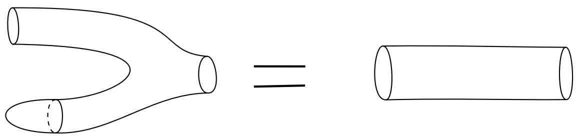
Associativity follows from the following picture.



□

Lemma G.3.3. *(Cap as the Identity)*
The product defined by $\Sigma_{0,3}$ *has unit* $Z(\Sigma_{0,1}, \varnothing, id_{S^1}) \in Z(S^1)$.

Proof. This follows from the fact that $Z(\Sigma_{0,2}, Id_{(S^1)*}, Id_{S^1}) = id_{Z(S^1)}$ and the following picture.



□

Definition: (Frobenius Algebra's)
Let $(A, \cdot)$ be a R-algebra. Let $\langle -, - \rangle : A \otimes A \to R$ be a non-degenerate bilinear form. If we have for $a, b, c \in A$ we have

$$\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle$$

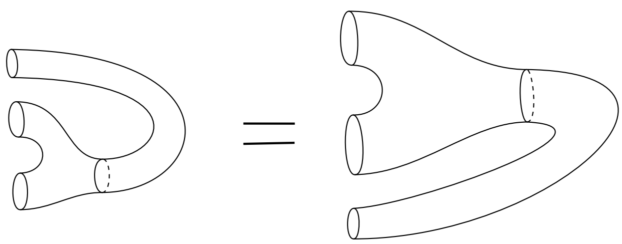
then we call $(A, \cdot, \langle -, - \rangle)$ a **Frobenius algebra**.

Theorem G.3.4. *(2d-TQFTs and Frobenius Algebras)*
The association

$$Z \mapsto (Z(S^1), Z(\Sigma_{0,3}, id_{(S^1)*} \times id_{(S^1)*}, id_{S^1}), Z(\Sigma_{0,2}, id_{(S^1)*} \times id_{(S^1)*}, \varnothing))$$

is a bijection between commutative Frobenius algebra's and $(1+1)$*-dimensional TQFTs.*

Proof. We have show that every $(1+1)$-dimensional TQFT defines a commutative algebra. Notice that it is a Frobenius algebra by the following picture.



To then see that every Frobenius algebra defines a $(1+1)$-dimensional TQFT one must consider trinion decompositions. We can see that using the product we can define a TQFT given a trinion decomposition then one must check that this is independent of the decomposition. This fact reduces to the identities defining Frobenius algebras. $\square$

> **Remark:** This theorem can be generalised to an association between symmetric monoidal functors from $\underline{Cob_{1+1}}$ to a category $\mathcal{C}$ and the category of commutative Frobenius objects in $\mathcal{C}$. That is
>
> $$\underline{cFrob(\mathcal{C})} \cong \left[\underline{Cob_{1+1}}, \mathcal{C}\right]_{SymMon}$$
>
> For this more general discussion see [Koc04].

Theorem G.3.5. *(Classification Formula for Semi-Simple* $(1+1)$*-dimensional TQFTs) [Law96]*
Let Z *be a* $(1+1)$*-dimensional TQFT over* $\mathbb{K}$ *with* $(V, \cdot, \langle -, - \rangle)$ *the associated Frobenius algebra over* $\mathbb{K}$*. If there exists a basis* $v_1, ..., v_m$ *for* V *such that* $v_i \cdot v_j = \delta_{ij} v_i$ *then the Frobenius algebra is determined up to isomorphism by* $\langle v_i, v_i \rangle = h_i$ *and we then have*

$$Z(\Sigma_{g,n})(v_{i_1}, ..., v_{i_n}) = \delta_{i_1 \dots i_n} h_{i_1}^{2(2g-2+n)} h_{i_1}^{-3g+3-n} = \delta_{i_1 \dots i_n} h_{i_1}^{g-1+n}$$

Proof. Taking a pair of pants decomposition we have $2g-2+n$ pairs of pants and $3g-3+n$ cylinders gluing the pairs of pants together. For each pair of pants we get a factor of $h_{i_1}^2$ and for each gluing cylinder we get a factor of $h_{i_1}^{-1}$. $\square$

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