Intro to quantum modularity:

Modular forms are f^n with many symetries.

Quantum modular forms don't satisfy these symetries but the failure of these lead to improved f^ns.

Defn: A holomorphic f^n $f : \mathfrak{h} \to \mathbb{C}$ is called a modular form of Γ (a finite index subgroup of $SL_2\mathbb{Z}$) and weight $k \in \mathbb{Z}$ if for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \qquad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

& $f(\tau)$ is bounded as $\mathrm{Im}(\tau) \to \infty$

Ex: $$G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m + n\tau)^{2k}}$$

$\Gamma = SL_2\mathbb{Z}$ weight $2k$.

Rmk: If f is a modular form of wieght $2k$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) d\left(\frac{a\tau+b}{c\tau+d}\right)^{\otimes k} = f(\tau)\, d\tau^{\otimes k}$$

give forms on $\Gamma \backslash \mathfrak{h} \cup \mathbb{P}^1\mathbb{Q}$.

Prop: If $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma$ then a modular form of Γ satisfies the equation

$$f(\tau+m) = f(\tau).$$

With the boundedness & holomorphicity we see that f has a Fourier series

$$f(\tau) = \sum_{k=0}^{\infty} c_k q^{\frac{k}{m}} \quad \text{where } q = e(\tau) = e^{2\pi i \tau}$$

Notation: We often write $q^{\frac{1}{m}} = e\left(\frac{\tau}{m}\right)$ & $f(q) = f(\tau)$.

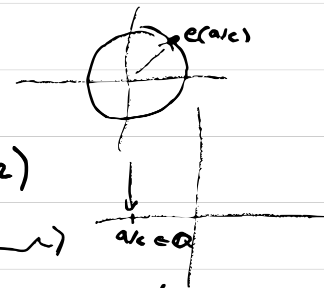
Ex: $$G_{2k}(q) = -2\zeta(2k)\frac{4k}{B_{2k}}\left(\frac{-B_{2k}}{4k} + \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n} \right)$$

$\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$

$$= -2\zeta(2k)\frac{4k}{B_{2k}}\left(-\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \right)$$

Asymptotics

Modular forms have well behaved asymptotics. As $\tau \to i\infty$



$$G_{2k}\left(\frac{a\tau+b}{c\tau+d}\right) \sim (c\tau+d)^k\, 2\zeta(2k)$$

$$\overset{\shortparallel}{G_{2k}\left(\frac{a}{c} - \frac{1}{c(c\tau+d)}\right)} \rightsquigarrow$$

There are more interesting examples.

Ex: (Rogers-Ramanujan f^n)

$$G(q) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k} \qquad H = \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}$$

where $(x;q)_k = \prod_{j=0}^{k-1} (1-xq^j)$. Consider

$S_\gamma(\tau) = \tilde{f}\, j^{-1} f^{-1}$

R.R. id.

$$g(q) = \begin{pmatrix} q^{-1/60} G(q) \\ q^{11/60} H(q) \end{pmatrix} = \cancel{\begin{pmatrix} q^{-1/60} (q;q^5)_\infty (q^4;q^5)_\infty \\ q^{11/60} (q^2;q^5)_\infty (q^3;q^5)_\infty \end{pmatrix}}$$

Then

$$g(\tau+1) = \begin{pmatrix} e(-1/60) & 0 \\ 0 & e(11/60) \end{pmatrix} g(\tau)$$

$$g(-1/\tau) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin(\frac{2\pi}{5}) & \sin(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) & -\sin(\frac{2\pi}{5}) \end{pmatrix} g(\tau)$$

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = SL_2\mathbb{Z}$$

Therefore, $g\left(\frac{a\tau+b}{c\tau+d}\right) = \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot g(\tau)$

for some $\rho : SL_2\mathbb{Z} \to GL_2\mathbb{C}$.
$\ker\rho$ is finite index

Therefore as $\tau \to i\infty$

$$g\left(\frac{a\tau+b}{c\tau+d}\right) \sim \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e(-\tau/60) \\ e(11\tau/60) \end{pmatrix}$$

eg $g(-\frac{1}{\tau}) \sim \frac{2}{\sqrt{5}}\begin{pmatrix} \sin 2\pi/5 \\ \sin \pi/5 \end{pmatrix} e(-\tau/60)$

Therefore the asymptotics of g determine ρ.

<u>Mock Θ-fn</u>

Ramanujan studied many q-hyp. examples like these. For example,

$$f(q) = \sum_{k=0}^{\infty} q^{k^2}/(-q;q)_k.$$

This fn is an example of what he called a mock theta fn.

Their asymptotics are like modular forms. For example, as $\tau \to i\infty$

$$f\left(\frac{\tau}{2\tau+1}\right) \sim e\left(\frac{-1}{60}\left(\tau+\frac{1}{2}\right)\right)\sqrt{\tau+\frac{1}{2}}\left(-\frac{15}{8}-\frac{5\sqrt{5}}{8}\right)$$
$$\times\, e\left(\frac{-1}{60}\,\frac{1}{2(2\tau+1)}\right)$$

to all orders

This looks like a modular form.
For a modular form we can subtract the RHS ~~a~~ to ~~get~~ find a similar asymptotic behavior at subleading.
Here we find

$$f\left(\frac{\tau}{2\tau+1}\right) - \mathrm{RHS} \sim 2 + 4\,\frac{-2\pi i}{2(2\tau+1)} + 36\left(\frac{-2\pi i}{2(2\tau+1)}\right)^2 + \dots$$

These asymptotics can't come from a modular form.

<u>Rmk</u>: What is this series

$$2 + 4h + 36h^2 + \dots \quad ?$$

$$-2\sum_{k=0}^{\infty} q^{k(k+3)/2+1}(-q;q)_k$$

sub $q=-e^h$ to find this expansion.

Another example of Ramanujan is

$$f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}=\frac{2}{(q;q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n}$$

Zwegers realized the expression on the right had some explicit modular property.
Letting

$$L(x,\lambda;q)=\frac{x^{\frac{1}{2}}q^{\frac{1}{8}}}{(qx;q)_\infty(x^{-1};q)_\infty(q;q)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k\frac{q^{k(k+1)/2}\lambda^k}{1-q^k\lambda x}$$

he proves

Thm

$$\sqrt{\tfrac{i}{\tau}}\,e\left(\tfrac{z^2}{2\tau}\right)L\left(e\left(\tfrac{z}{\tau}\right),e\left(\tfrac{w}{\tau}\right);e\left(-\tfrac{1}{\tau}\right)\right)-L(e(z),e(w);e(\tau))$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e(x^2\tau/2+ixz)}{\cosh(\pi x)}dx$$

Laplace transform!

We see the failure of modularity gives a fn we can be extended to $\tau \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. This is an example of quantum modularity.

Rmk: Zwegers found a real analytic fn which satisfied the same eqn. Therefore their difference gave a real analytic modular form. Mock theta fn were realized to be the holomorphic part of such fn.

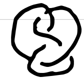
Quasd

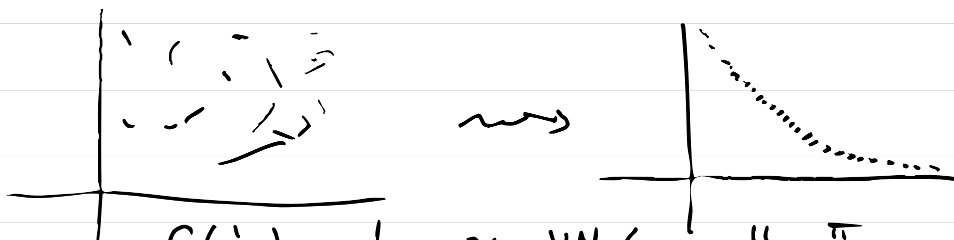
"**Def 1**": A quantum modular form is a f $f: \mathbb{Q} \to \mathbb{C}$ s.t

$$f\left(\frac{a\tau+b}{c\tau+d}\right) - (c\tau+d)^k f(\tau)$$

is better eg real analytic.

$\underline{\underline{\text{Ex}}}$: $f(\frac{r}{s}) = \log \sum_{k=0}^{s} (e(\frac{r}{s}); e(\frac{r}{s}))_k (e(\frac{-r}{s}); e(\frac{-r}{s}))_k$

This fn comes from the knot .



$$f(\tfrac{1}{N}) \sim \frac{1}{3^{1/4}} N^{3/2} e^{VN} \left(1 + \frac{11}{36\sqrt{3}} \frac{\pi}{N} + \cdots \right)$$

V transcendental period associated with $\mathbb{Q}(\sqrt{-3})$

This was original defn but recently there have been greater understanding of examples.

$\underline{\underline{\text{Defn}}}$: f is called a quantum modular form if $\exists\, \Omega$ a cocycle on $SL_2\mathbb{Z}$ with $\Omega_\gamma(\tau)$ analytic on $\mathbb{C} \setminus \frac{1}{c}\mathbb{R}_{\le -d}$ s.t.

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \Omega_\gamma(\tau) f(\tau) (c\tau+d)^k$$

We say f has cocycle Ω and weight k.

$$\Omega_{\gamma_1 \cdot \gamma_2}(\tau) = \Omega_{\gamma_1}(\gamma_2 \tau) \Omega_{\gamma_2}(\tau).$$

f could be a fn from $\mathbb{Q}$, $\mathfrak{h}$, $\mathfrak{h}^-$

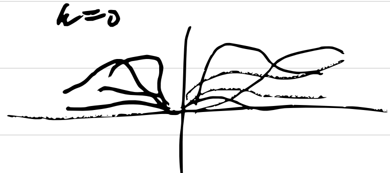
In fact we often have an

$$f : \mathfrak{h} \cup \mathbb{Q} \cup \mathfrak{h}^- \longrightarrow \mathbb{C}^N.$$ connected by Ω.

Remark

Ω is not a priori a coboundary as domain of analyticity is different.

Ex: $J : \mathbb{Q} \to \mathbb{C}^3$ $\quad J(\xi) = \sum_{k=0}^{\infty} (q;q)_k (q^{-1};q^{-1})_k \begin{pmatrix} 1 \\ q^k \\ q^{2k} \end{pmatrix}$. $\quad q = e(\xi)$

Ω 

Ex: Vector valued modular forms have cocycle $\Omega_\gamma(\tau) = \rho(\gamma)$ for some rep $\rho : SL_2\mathbb{Z} \to GL_N(\mathbb{C})$.

$0 \longrightarrow$ "modular forms" $\to$ "quantum modular forms" $\to$ "cocycles"

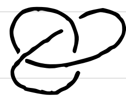
Prop: Suppose f, g are two bases of quantum modular forms with cocycle Ω then

$g^{-1} f$ is a matrix of modular forms

proof $g\left(\frac{a\tau+b}{c\tau+d}\right)^{-1} f\left(\frac{a\tau+b}{c\tau+d}\right) = \Delta(c\tau+d)^{-1} g(\tau)^{-1} f(\tau)\, \Delta'(c\tau+d)$

Remark

Therefore we see that upto modular forms everything is determined by the cocycle Ω.

Ex:  $J(q) = q^{\frac{1}{24}} \sum_{k=0}^{\infty} (q;q)_k$ $\qquad J: \mathbb{Q} \to \mathbb{C}$.

$\left(\frac{12}{n}\right) \leftarrow$ Kronecker symbol.

$\left(\frac{12}{n}\right) =$ $\qquad \tilde{\eta}(q) = \sum_{n=1}^{\infty} n \chi(n) q^{\frac{n^2}{24}}$

$\eta(q) = \sum_{n=1}^{\infty} \chi(n) q^{\frac{n^2}{24}}$

Thm (Zagier) $J(q) \text{ "=" } -\frac{1}{2}\tilde{\eta}(q)$

i.e. asymp. as $q \sim \zeta$ root of unity agree to all orders.

Ex: $J(e^{-t}) = \sum_{n=0}^{\infty} \frac{T_n}{n!}\left(\frac{t}{24}\right)^n \sim -\frac{1}{2}\tilde{\eta}(q)$

where $\sum_{n=0}^{\infty} \frac{T_n}{(2n+1)!} x^{2n+1} = \frac{\sin 2x}{2\cos 3x}$.

$f(q) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\tilde{\eta}(q) & \eta(q) \end{pmatrix}$ or $f(q) = \begin{pmatrix} 1 & 0 \\ J(q) & \omega(q) \end{pmatrix}$

$|q| < 1$ $\qquad q^r = 1$

$$f(\tau)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} f\left(-\tfrac{1}{\tau}\right)$$ is an analytic fn on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$

is the Borel resummation of $J(e^{-t})$. Thm [Costin-Garoufalidis]

<u>Moral</u>: Cocycles of quantum modular forms give resummation of associated asym series.