Resurgence

formal divergent power series $\tilde{\Phi} := \sum_{n=0}^{\infty} a_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$

↳ 0-radius of conv.
" a_n grow faster than A^n "

$a_n \sim n!\, A^n$ 1- Gevrey

- perturbation theory
- asymptotics $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \sim e^{-z} z^{z-\frac{1}{2}} \sqrt{2\pi}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots\right.$
- solving irr. sing. ODEs $\Phi' = -\Phi + \frac{1}{z}$

GOAL: find non-perturbative contributions
"Resum" the formal solution to get an analytic one

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① Borel transform " remove divergence "

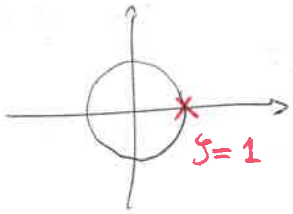
formal $\quad \mathcal{B}_\xi[z^{-n-1}] := \frac{\xi^n}{n!}$ + linearity

$$\mathcal{B}_\xi : \underset{\substack{\cap \\ z^{-1}\mathbb{C}[[z^{-1}]]_1}}{\sum_{n=0}^{\infty} a_n z^{-n-1}} \longrightarrow \underset{\substack{\cap \\ \mathbb{C}\{\xi\}}}{\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}} =: \hat{\varphi}$$ finite radius of conv.

e.g. $\sum_{n=0}^{\infty} n!\, z^{-n-1} \xrightarrow{\mathcal{B}_\xi} \sum_{n=0}^{\infty} n! \frac{\xi^n}{n!} = \sum_{n=0}^{\infty} \xi^n \qquad |\xi| < 1$

- study analytic continuation of $\hat{\varphi}$

e.g. $\hat{\varphi} = \frac{1}{1-\xi}$ sum of $\tilde{\varphi}$



z-frequency

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\tilde{\Phi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1 \arrow[r, shift right, "\text{BL sum}"'] \arrow[d, "\mathcal{B}_\zeta"'] & \hat{\Phi} \arrow[l, shift right, "\text{1-Gevrey ae}"'] \\
\tilde{\phi} \in \mathbb{C}\{\zeta\} \arrow[r, "\text{sum}"] & \hat{\phi} \arrow[u, "\mathcal{L}_{\zeta,0}"']
\end{tikzcd}$$

formal

ζ-position (Borel plane)

analytic away from singularities

to go back to z-plane we want inverse of $\mathcal{B}_\zeta$ but ANALYTIC

② Laplace

$$\mathcal{L}_{\zeta,0} f := \int_0^{+\infty} e^{-z\zeta} f \, d\zeta \qquad f \text{ integrable} \quad |f(\zeta)| \leq A e^{c_0|\zeta|} \quad \zeta \geq 1$$

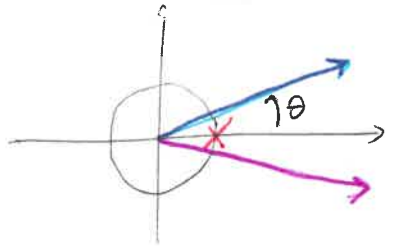
$$\mathcal{L}_{\zeta,0}\left[\frac{\zeta^n}{n!}\right] = z^{-n-1} \qquad (\text{Borel is formal inverse of Laplace})$$

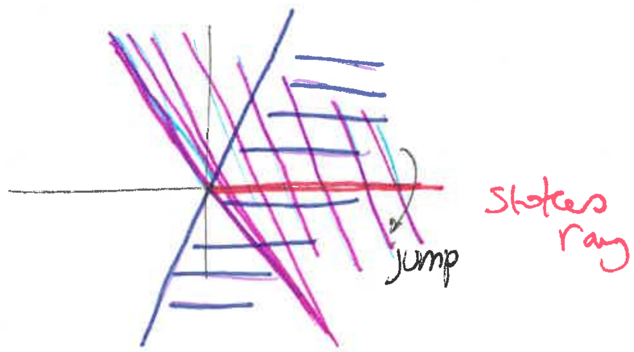
Thm: $\mathcal{L}_{\zeta,0} f$ in holom. in $\{\operatorname{Re} z > c_0\}$ [draw diagram]

Def: Borel-Laplace sum

- change direction : $\mathcal{L}^\theta_{\zeta,0} f := \int_0^{+\infty e^{i\theta}} e^{-z\zeta} f \, d\zeta$ holom. in $\{\operatorname{Re} z e^{-i\theta} > c_0$

e.g.





$$\hat{\Phi}_+ = \mathcal{L}^{\theta}_{\zeta,0} \frac{1}{1-\zeta}$$

$$\hat{\Phi}_- = \mathcal{L}^{-\theta}_{\zeta,0} \frac{1}{1-\zeta} \qquad \text{two analytic functions}$$

Having singularity at $\zeta = 1 \Rightarrow \hat{\Phi}_+ \neq \hat{\Phi}_-$ where both defined

$$\hat{\Phi}_+ - \hat{\Phi}_- = \oint e^{-z\zeta} \frac{1}{1-\zeta} d\zeta = e^{-z} \cdot 2\pi i \quad \text{jump!} \quad \to \text{STOKES PHENOMENA}$$

Rmk: asymptotically $\hat{\Phi}_+, \hat{\Phi}_-$ have same behaviour Re $z \to +\infty$.

"Jump" is non perturbative, subleading contribution

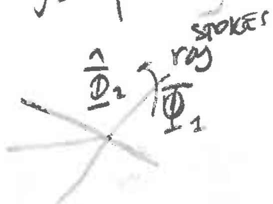
$\boxed{\text{Thm}}$ (inverse of BL sum)

$\hat{\Phi}$ is uniform Gevrey asympt. to $\tilde{\Phi}$ in Re $z > c$ $c > c_0$.

$$\left| \hat{\Phi}(z) - \sum_{n=0}^{N} a_n z^{-n-1} \right| \leq C A^N N! \, |z|^{-N-2}$$

Rmk: if you are familiar with ODEs, STOKES PHENOMENA is common when there are irregular singularities.
Monodromy data of your system corresponds to matrices of STOKES constants.

Summary: $\tilde{\Phi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ divergent $\overset{\text{BL sum}}{\rightsquigarrow}$ $\left\{ \hat{\Phi}_j \right.$ hol. functions in different sectors $\left| \text{ jumps} \right\}$



Resurgence computes "jumps" from the study of $\hat{\phi}$ in Borel plane

↳ singularities

↳ analytic continuation

e.g. $\hat{\phi} = \frac{1}{1-\zeta}$ Jump: $e^{-z \cdot 1} \cdot 2\pi i$

↳ singularity

Res $\hat{\phi}$ ~ (analytic continuation) monodromy

Def: $\tilde{\Phi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1$ is resurgent series if $\hat{\phi} := \mathcal{B}\tilde{\Phi}$ has endless analytic continuation away from $\Omega \subseteq \mathbb{C}$.

$\forall L > 0$ $\exists \Omega_L \subseteq \mathbb{C}$ finite s.t. $\hat{\phi}$ can be analytically continued along any paths starting near $\zeta = 0$ and of lenght $\leq L$ in $\mathbb{C} \setminus \Omega_L$.

Ω can be countable,

NOT RESURGENT: $\sum_{n=0}^{\infty} a_n n! \, z^{-n-1}$ $a_n = \begin{cases} -1 & n = 2^k \\ 1 & \text{otherwise} \end{cases}$ $\sum_{k=0}^{\infty} \zeta^{2^k}$ $\zeta_m = e^{\frac{2\pi i}{2^m}}$ $|\zeta| < 1$

Simple resurgent functions

Ω set of singularities, $\omega \in \Omega$

$$\hat{\phi}(\zeta+\omega) = \frac{C_\omega}{2\pi i \zeta} + \frac{S_\omega}{2\pi i} \log \zeta \, \tilde{\phi}_\omega(\zeta) + \text{h.f.} \qquad \text{near } \zeta = 0$$

$C_\omega, S_\omega \in \mathbb{C}$, $\tilde{\phi}_\omega(\zeta) \in \mathbb{C}\{\zeta\}$

– what is the revelent information?

e.g. 1 $\tilde{\Phi}$ = asymptotics of $\frac{\Gamma(z)\, e^{z}\, z^{-z+\frac{1}{2}}}{\sqrt{\pi}}$ as $\operatorname{Re} z \to \infty$

$$\Omega = 2\pi i \mathbb{Z}, \quad \hat{\phi}(\zeta + 2\pi i) = \frac{1}{2\pi i} \log(\zeta)\, \underbrace{\tilde{\phi}_{2\pi i}(\zeta)}_{= \tilde{\phi}(\zeta)} + \text{h.f}$$

same as $\mathcal{B}\tilde{\Phi}$

(see Maxim's talk from this summer)

2 $\left[\partial_z^2 - 1 + \frac{1}{z}\partial_z - \frac{1}{9z^2}\right]\phi = 0$

↳ formal solutions $\tilde{\Phi}_{\pm 1}(z) = e^{\pm z} z^{-1/2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{2^n n!} (\mp z)^{-n-1}$

$$\hat{\phi}_1(\zeta) = {}_2F_1\left(a,b,c;-\tfrac{\zeta}{2}\right) \Rightarrow \hat{\phi}_1(\zeta+2) = \frac{1}{2\pi i}\log(\zeta)\, \tilde{\phi}_{-1}(\zeta) + \text{h.f.}$$

$$\hat{\phi}_{-1}(\zeta) = {}_2F_1\left(a,b,c;\tfrac{\zeta}{2}\right) \Rightarrow \hat{\phi}_{-1}(\zeta-2) = \frac{-1}{2\pi i}\log(\zeta)\, \tilde{\phi}_1(\zeta) + \text{h.f.}$$

new Series!

idea of resurgence: functions near singularities reproduce themselves, "they resurge near their singularities" like in 1

Écalle introduced Alien calculus, and formalism of singularities to compute $(C_\omega, S_\omega, \tilde{\phi}_\omega)$

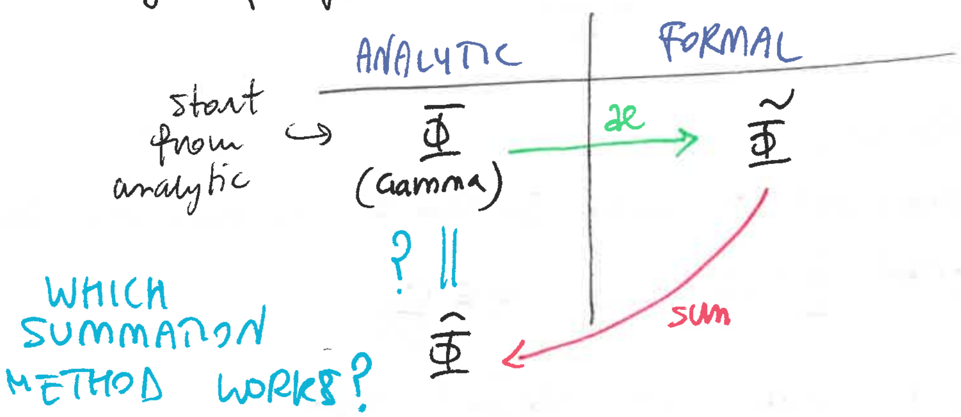
We refer to $(C_\omega, S_\omega, \tilde{\phi}_\omega \mid \omega \in \Omega)$ as resurgent ~~data~~ structure

How do you check something is resurgent?

- knowing a close formula for the coefficients
  - ↳ know the sum (like hypergeometric)
  - ↳ HADAMARD PRODUCT to compute the sum of $\tilde{\phi}$
- knowing only $a_1, a_2, \dots a_N$, no close formula
  - ↳ Borel + Padé numerically, at least to see the position of the singularities.
- the formal series comes from solving ODEs, PDEs, difference eq. you might know it's resurgent because of the problem you're solving (Borel, Laplace behaves well w/ ∂_z, T_c, products, ...)

Back to summability

- Borel-Laplace $\tilde{\Phi} \in z^{-1}\mathbb{C}[[z^{-1}]]_1 \longrightarrow \Phi$ analytic
- change perspective

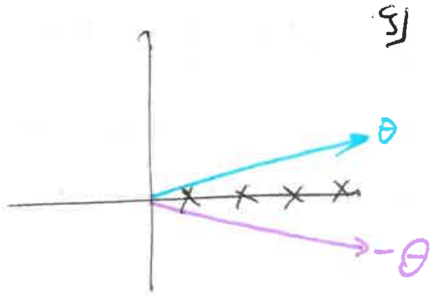


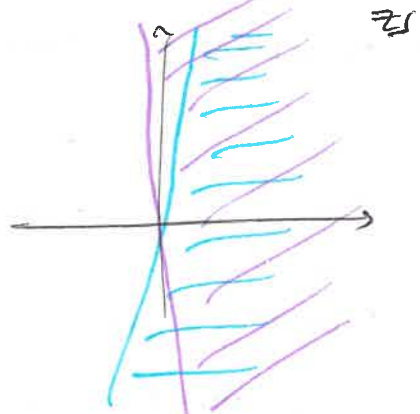
(for $\Gamma(z)$, Borel-Laplace works)

THM (Nevanlinna) If Φ is uniform Gevrey, asymptotics ~~sum~~ in a sector of opening angle π, to $\tilde{\Phi}$, then $\Phi = \mathcal{L} \circ \mathcal{B}\, \tilde{\Phi}$.

Def (median resummation)

$$S^{med} = \frac{\mathcal{L}^{\theta} + \mathcal{L}^{-\theta}}{2}$$





think of S^{med} as ~~along the ray~~ an approximation of $\mathcal{L}$ along ray of singularities.

S^{med} is defined for $\mathrm{Re}\, z > 0$.

rewrite

$$S^{med} = \begin{cases} \mathcal{L}^{\theta} - \frac{disc}{2} \\ \mathcal{L}^{-\theta} + \frac{disc}{2} \end{cases} = \begin{cases} \mathcal{L}^{\pi/2} - \frac{disc}{2} \\ \mathcal{L}^{-\pi/2} + \frac{disc}{2} \end{cases}$$

$disc := \mathcal{L}^{\theta} - \mathcal{L}^{-\theta}$

deform the contour below, and collect all the singularities

Conj [Costin-Garoufalidis]

If Φ is WRT invariant of closed 3-manif./Kashaev inv. for knots then $\Phi(z) = S^{med}\, \tilde{\Phi}(-\frac{1}{z})$ at $z \in \frac{\mathbb{Q}}{2\pi i}$

Thm [Costin-Garoufalidis] true for trefoil knot.

Rmk: $\Phi(z) \neq \mathcal{L}^{\pi/2}\mathcal{B}\tilde{\Phi}$ $\quad \mathrm{Im}\, z > 0$

$\Phi(z) = \mathcal{L}^{\pi/2}\mathcal{B}\tilde{\Phi} - \boxed{\frac{disc}{2}}$ at $z \in \frac{\mathbb{Q}}{2\pi i}$ // rationals

corrections coming from resurgent structure

Here there is something interesting going on, I think it's related with QUANTUM MODULARITY ...