STATISTICAL REGULARITY OF APOLLONIAN GASKETS

XIN ZHANG

ABSTRACT. Apollonian gaskets are formed by repeatedly filling the gaps between three mutually tangent circles with further tangent circles. In this paper we give explicit formulas for the the limiting pair correlation and the limiting nearest neighbor spacing of centers of circles from a fixed Apollonian gasket. These are corollaries of the convergence of moments that we prove. The input from ergodic theory is an extension of Mohammadi-Oh's Theorem on the equidisribution of expanding horospheres in infinite volume hyperbolic spaces.

1. INTRODUCTION

1.1. Introduction to the problem and statement of results.

Apollonian gaskets, named after the ancient Greek mathematician Apollonius of Perga (200 BC), are fractal sets formed by starting with three mutually tangent circles and iteratively inscribing new circles into the curvilinear triangular gaps (see Figure 1).
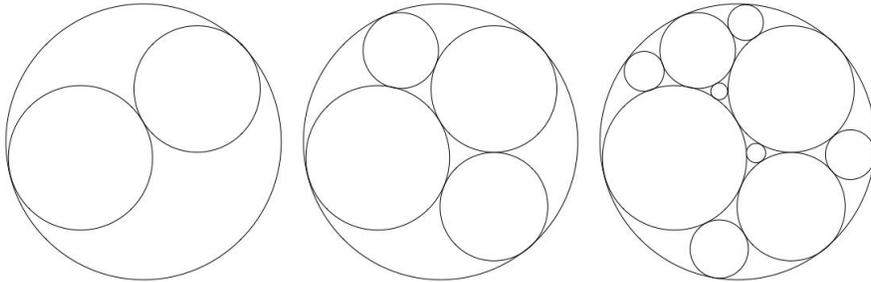
FIGURE 1. Construction of an Apollonian gasket

The last 15 years have overseen tremendous progress in understanding the structure of Apollonian gaskets from different viewpoints, such as number theory and geometry [16], [15], [10], [11], [20], [25]. In the geometric direction, generalizing a result of [20], Hee Oh and Nimish Shah proved the following remarkable theorem concerning the growth of circles.

Place an Apollonian gasket $\mathcal{P}$ in the complex plane $\mathbb{C}$. Let $\mathcal{P}_t$ be the set of circles from $\mathcal{P}$ with radius greater than e^{-t} and let $\mathcal{C}_t$ be the set of centers from $\mathcal{P}_t$. Oh-Shah proved:

Theorem 1.1 (Oh-Shah, Theorem 1.6, [25]). *There exists a finite Borel measure v supported on $\overline{\mathcal{P}}$, such that for any compact region $E \subset \mathbb{C}$ with boundary ∂E empty or*

piecewise smooth (see Figure 2), the cardinality $N(E, t)$ of the set $\mathcal{C}_t \cap E$, satisfies

$$\lim_{t \to \infty} \frac{N(E, t)}{e^{\delta t}} = v(E),$$

where $\delta \approx 1.305688$ [22] is the Hausdorff dimension of any Apollonian gasket.
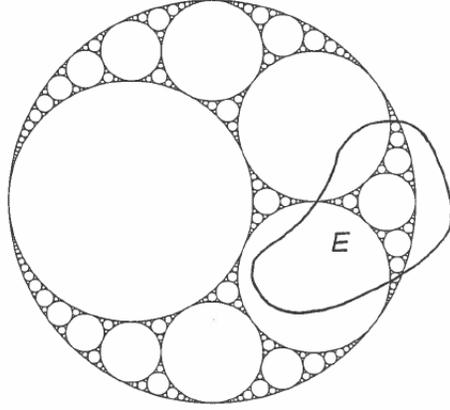
FIGURE 2. A region E with piecewise smooth boundary

Theorem 1.1 gives a satisfactory explanation on how circles are distributed in an Apollonian gasket in large scale. In this paper we study some questions concerning the fine scale distribution of circles, for which Theorem 1.1 yields little information. For example, one such question is the following.

Question 1.2. *Fix $\xi > 0$. How many circles in $\mathcal{P}_t$ are within distance ξ/e^t of a random circle in $\mathcal{P}_t$?*

Here by distance of two circles we mean the Euclidean distance of their centers. Question 1.2 is closely related to the pair correlation of circles. In this article, we study the pair correlation and the nearest neighbor spacing of circles, which concern the fine structures of Apollonian gaskets. In particular, Theorem 1.3 gives an asymptotic formula for one half of the expected number of circles in Question 1.2, as $t \to \infty$.

Let $E \subset \mathbb{C}$ be an open set with ∂E empty or piecewise smooth as in Theorem 1.1, and with $v(E) > 0$ (if $E \cap \overline{\mathcal{P}} \neq \emptyset$ then $v(E) > 0$). This is our standard assumption for E throughout this paper. The pair correlation function $P_{E,t}$ on the growing set $\mathcal{C}_t$ is defined as

$$P_{E,t}(\xi) := \frac{1}{2\#\{\mathcal{C}_t \cap E\}} \sum_{\substack{p,q \in \mathcal{C}_t \cap E \\ q \neq p}} \mathbf{1}\{e^t |p - q| < \xi\}, \tag{1.1}$$

where $\xi \in (0, \infty)$ and $|p - q|$ is the Euclidean distance between p and q in $\mathbb{C}$. We have a factor $1/2$ in the definition (1.1) because we only want to count each pair of points once.

For any $p \in \mathcal{C}_t$, let $d_t(p) = \min\{|q - p| : q \in \mathcal{C}_t, q \neq p\}$. The nearest neighbor spacing function $Q_{E,t}$ is defined as

$$Q_{E,t}(\xi) = \frac{1}{\#\{\mathcal{C}_t \cap E\}} \sum_{p \in \mathcal{C}_t \cap E} \mathbf{1}\{e^t d_t(p) < \xi\} \tag{1.2}$$

For simplicity we abbreviate $P_{E,t}, Q_{E,t}$ as P_t, Q_t if $E = \mathbb{C}$. It is noteworthy that in both definitions (1.1) and (1.2), we normalize distance by multiplying by e^t. The reason can be seen in two ways. First, Theorem 1.1 implies that a random circle in $\mathcal{C}_t$ has radius $\asymp e^{-t}$, so a random pair of nearby points (say, the centers of two tangent circles) from $\mathcal{C}_t$ has distance $\asymp e^{-t}$, thus e^{-t} is the right scale to measure the distance of two nearby points in $\mathcal{C}_t$. The second explanation is more informal: if N points are randomly distributed in the unit interval $[0, 1]$, then a random gap is of the scale N^{-1}; more generally, if N points are randomly distributed in a compact manifold of dimension n, the distance between a random pair of nearby points should be of the scale $N^{-1/n}$. In our situation, as $t \to \infty$, the set $\mathcal{C}_t$ converges to $\overline{P}$, where $\overline{\mathcal{P}}$ has Hausdorff dimension $\delta \approx 1.305688$. From Theorem 1.1, we know that $\#\mathcal{C}_t \asymp e^{\delta t}$, so our scaling e^{-t} agrees with the heuristics that the distance between two random nearby points in $\mathcal{C}_t$ should be $(e^{\delta t})^{-\frac{1}{\delta}} = e^{-t}$.

Before stating our main results, we introduce terminology. It is convenient for us to work with the upper half-space model of the hyperbolic 3-space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{z + r\mathbf{j} : z = x + y\mathbf{i} \in \mathbb{C}, r \in \mathbb{R}\}.$$

We identify the boundary $\partial \mathbb{H}^3$ of $\mathbb{H}^3$ with $\mathbb{C} \cup \{\infty\}$. For $q = x + y\mathbf{i} + r\mathbf{j} \in \mathbb{H}^3$, we define $\Re(q) = x + y\mathbf{i}$ and $\Im(q) = r$.

Let $G = PSL(2, \mathbb{C})$ be the group of orientation-preserving isometries of $\mathbb{H}^3$. We choose a discrete subgroup $\Gamma < PSL(2, \mathbb{C})$ whose limit set $\Lambda(\Gamma) = \overline{P}$ such that Γ acts transitively on circles from $\mathcal{P}$. It follows from Corollary 1.3, [6] that Γ is geometrically finite.

Without loss of generality, we can assume that the bounding circle of $\mathcal{P}$ is $C(0, 1)$, where $C(z, r) \subset \mathbb{C}$ is the circle centered at z with radius r. Let $S \subset \mathbb{H}^3$ be the hyperbolic geodesic plane with $\partial S = C(0, 1)$, and $H < PSL(2, \mathbb{C})$ be the stabilizer of S.

As an isometry on $\mathbb{H}^3$, each $g \in PSL(2, \mathbb{C})$ sends S to a geodesic plane, which is either a vertical plane or a hemisphere in the upper half-space model of $\mathbb{H}^3$. We define continuous maps $\mathbf{q} : G \to \overline{\mathbb{H}^3}$, $\mathbf{q}_{\Re} : G \to \widehat{\mathbb{C}}$ as follows:

$$\mathbf{q}(g) := \begin{cases} \text{the apex of } g(S), & \text{if } \infty \notin g(\partial S), \\ \infty, & \text{if } \infty \in g(\partial S) \end{cases} \tag{1.3}$$

$$\mathbf{q}_{\Re}(g) := \begin{cases} \Re(\mathbf{q}(g)), & \text{if } \infty \notin g(\partial S), \\ \infty, & \text{if } \infty \in g(\partial S). \end{cases} \tag{1.4}$$

We further define a few subsets of $\mathbb{H}^3$. Let $B_\xi := \{z \in \mathbb{C} : |z| < \xi\}$ and let $B_\xi^* \subset \mathbb{H}^3$ be the "infinite chimney" with base B_ξ, where for any $\Omega \subset \mathbb{C}$,

$$\Omega^* := \{z + r\mathbf{j} : z \in \Omega, r \in (1, \infty)\}. \tag{1.5}$$

Let $\mathfrak{C}_\xi$ be the cone in $\mathbb{H}^3$:

$$\mathfrak{C}_\xi := \left\{ z + r\mathbf{j} \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{\xi}, 0 < r \leq 1 \right\}. \tag{1.6}$$

Now we can state our main theorems.

Theorem 1.3 (limiting pair correlation)**.** *For any open set $E \subset \mathbb{C}$ with ∂E empty or piecewise smooth, there exists a continuously differentiable function P independent of E, supported on $[c, \infty)$ for some $c > 0$, such that*

$$\lim_{t \to \infty} P_{E,t}(\xi) = P(\xi).$$

The derivative P' of P is explicitly given by

$$P'(\xi) = \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\substack{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H) \\ \mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi}} \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} d\mu_H^{\mathrm{PS}}(h).$$

Here $\Gamma_H := \Gamma \cap H$, and μ_H^{PS} is a Patterson-Sullivan type measure on H. Besides μ_H^{PS}, we will also encounter other conformal measures $\mu_N^{\mathrm{PS}}, w, m^{\mathrm{BR}}, m^{\mathrm{BMS}}$, which are built on the Patterson-Sullivan densities. The measure μ_N^{PS} is a Patterson-Sullivan type measure on the horospherical group $N := \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$, w is the pullback measure of μ_N^{PS} on $\mathbb{C}$ under the identification $z \to n_z$, and $m^{\mathrm{BR}}, m^{\mathrm{BMS}}$ are the Burger-Roblin, Bowen-Margulis-Sullivan measures. We will have a detailed discussion of these measures in Section 4.

See Figure 3 and Figure 5 for some numerical evidence for Theorem 1.3. Let $\mathcal{P}(\theta_1, \theta_2)$ be the unique Apollonian gasket determined by the four mutually tangent circles C_0, C_1, C_2, C_3, where $C_0 = C(0, 1)$ is the bounding circle, and C_1, C_2, C_3 are tangent to C_0 at $1, e^{\theta_1 \mathbf{i}}, e^{\theta_2 \mathbf{i}}$. Figure 3, Figure 4 and Figure 5 are based on the gasket $\mathcal{P}(\frac{1.8\pi}{3}, \frac{3.7\pi}{3})$. Figure 6 suggests that the limiting pair correlations for different Apollonian gaskets are the same. The reason is twofold. First, for a fixed gasket, the limiting pair correlation locally looks the same everywhere. Second, one can take any Apollonian gasket to any other one by a Möbius transformation, which locally looks like a dilation combined with a rotation, and it is an elementary exercise to check that the limiting pair correlation is invariant under these motions.
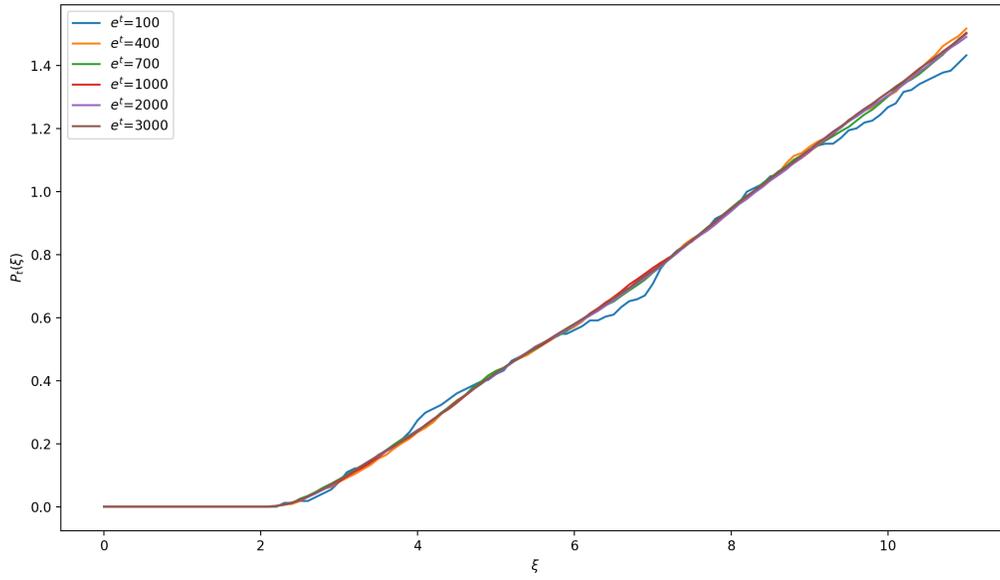
FIGURE 3. The plot for P_t with various t's
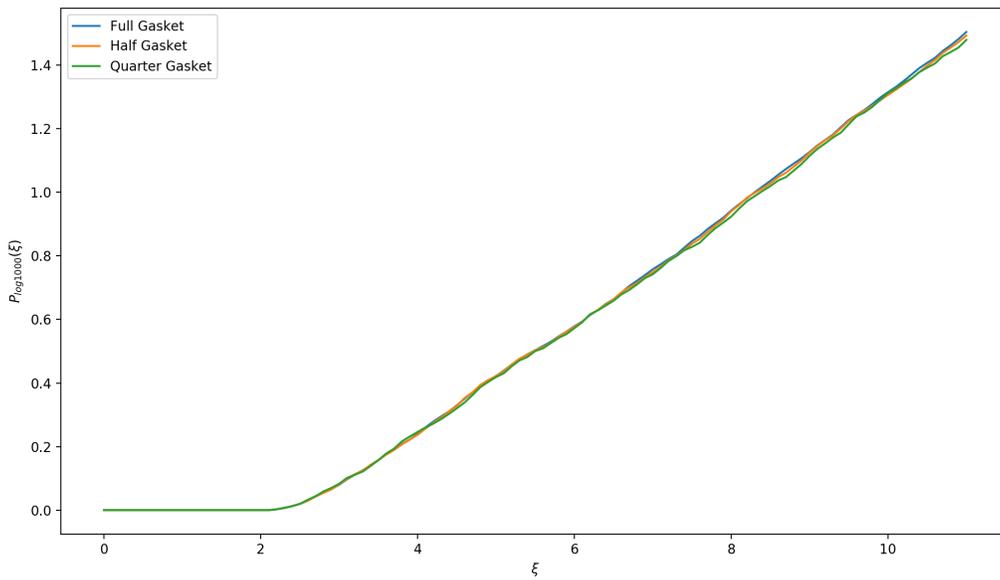
FIGURE 4. Pair correlation for the whole plane, half plane and the first quadrant
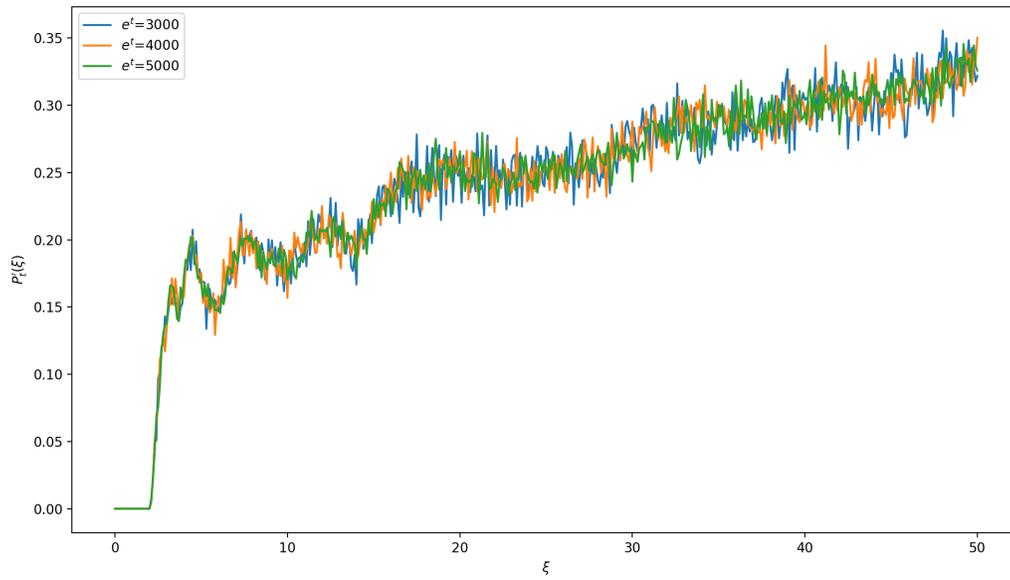
FIGURE 5. The empirical derivative $P_t'(\xi)$ for different t, with step=0.1
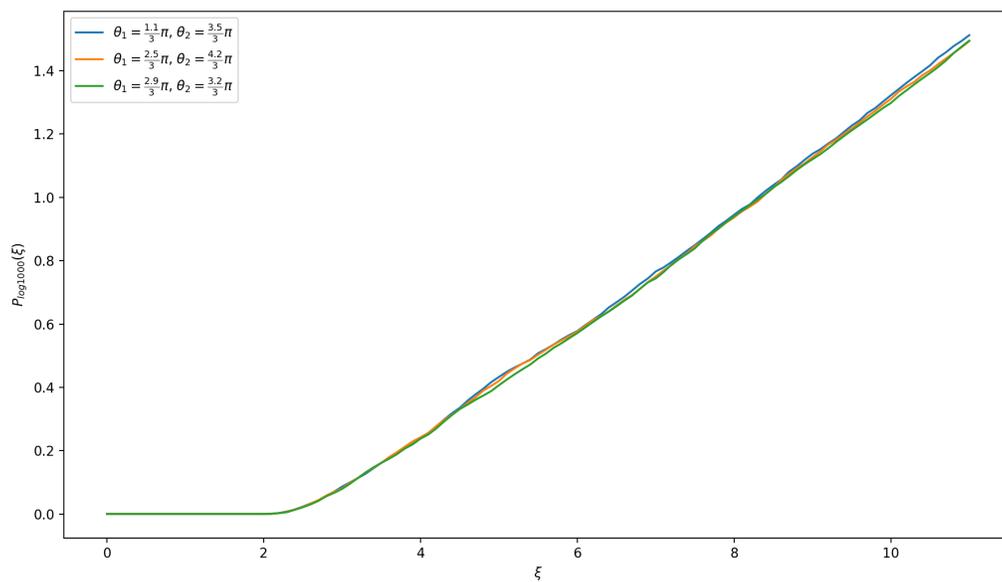
FIGURE 6. Pair correlation functions for different Apollonian gaskets

Theorem 1.4 (limiting nearest neighbor spacing)**.** *There exists a continuous function Q independent of E, supported on $[c, \infty)$ for some $c > 0$, such that*

$$\lim_{t \to \infty} Q_{E,t}(\xi) = Q(\xi). \tag{1.7}$$

The formula for Q is explicitly given by

$$Q(\xi) = 1 - \frac{\delta}{\mu_H^{PS}(\Gamma_H \backslash H)} \int_{\Gamma_H \backslash H} \int_0^\infty e^{-\delta t} \mathbf{1}\{\#\mathbf{q}(a_{-t}h^{-1}(\Gamma - \Gamma_H)) \cap B_\xi^* = 0\} dt d\mu_H^{PS}(h). \tag{1.8}$$

See Figure 7 for numerical evidence.
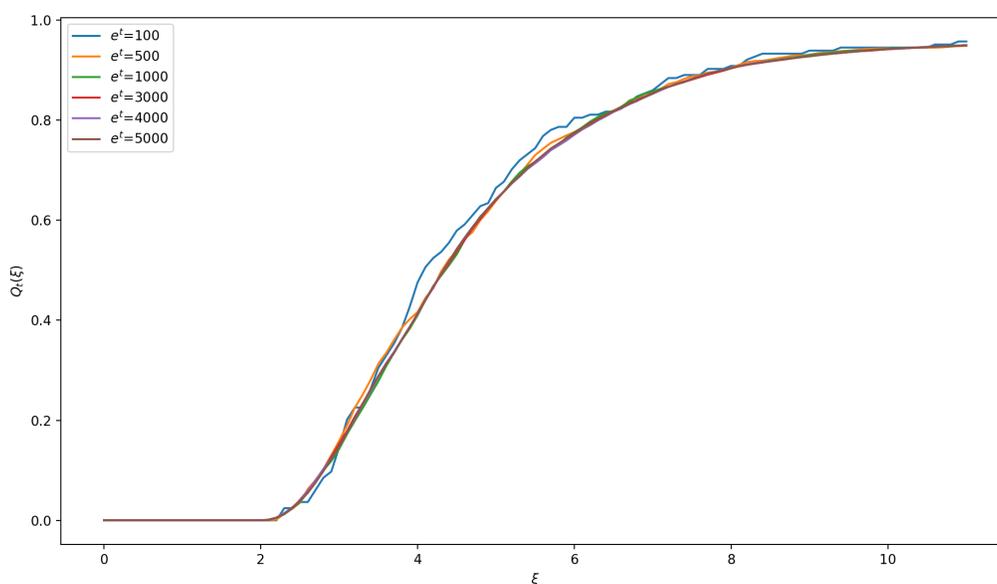
FIGURE 7. The nearest neighbor spacing function $Q_t(\xi)$ for various t's

Remark 1. Figure 7 suggests that Q should be differentiable. Unlike the limiting pair correlation, we have not been able to prove the differentiability of Q based on our formula for Q.

Both Theorem 1.3 and Theorem 1.4 follow from the convergence of moments (Theorem 1.5), which we explain now.

Let $\boldsymbol{\Omega} = \prod_{1 \leq i \leq k} \Omega_i \subset \mathbb{C}^k$, where $\Omega_i, 1 \leq i \leq k$ are bounded open subsets of $\mathcal{C}$ with piecewise smooth boundaries.

Let

$$\mathcal{B}_t(\Omega_i, z) := (e^{-t}\Omega_i + z) \cap \mathcal{C}_t,$$

and

$$\mathcal{N}_t(\Omega_i, z) := \#\mathcal{B}_t(\Omega_i, z).$$

Let $\mathbf{r} = \langle r_1, \ldots, r_k \rangle, \boldsymbol{\beta} = \langle \beta_1, \ldots, \beta_k \rangle$ be multi-indices, where $r_i \in \mathbb{Z}_{\geq 0}, \beta_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq k$, and at least one component of $\mathbf{r}, \boldsymbol{\beta}$ is nonzero. We want to understand the behaviors of the following two integrals

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathbf{1}\{\mathcal{N}_t(\Omega_i, z) = r_i\} \chi_E(z) dz \tag{1.9}$$

and

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathcal{N}_t(\Omega_i, z)^{\beta_i} \chi_E(z) dz, \tag{1.10}$$

as $t \to \infty$, where χ_E is the characteristic function for an open set $E \subset \mathbb{C}$ with no boundary or piecewise smooth boundary. Both (1.9) and (1.10) capture information about the correlation of centers.

Define functions $F_{\boldsymbol{\Omega},\mathbf{r}}, F_{\boldsymbol{\Omega}}^{\boldsymbol{\beta}}$ on G by

$$F_{\boldsymbol{\Omega},\mathbf{r}}(g) := \prod_{1 \leq i \leq k} \mathbf{1}\left\{\#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*) = r_i\right\} \tag{1.11}$$

$$F_{\boldsymbol{\Omega}}^{\boldsymbol{\beta}}(g) := \prod_{1 \leq i \leq k} \#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*)^{\beta_i} \tag{1.12}$$

We put inverse signs over g in the definitions (1.11) and (1.12) so that both $F_{\boldsymbol{\Omega},\mathbf{r}}$ and $F_{\boldsymbol{\Omega}}^{\boldsymbol{\beta}}$ are left Γ-invariant functions and can be thought of as functions on $\Gamma \backslash G$.

The following theorem holds:

Theorem 1.5 (convergence of moments). *With notation as above, we have*

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathbf{1}\{\mathcal{N}_t(\Omega_i, z) = r_i\} \chi_E(z) dz = \frac{m^{\mathrm{BR}}(F_{\boldsymbol{\Omega},\mathbf{r}}) w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)},$$

and

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathcal{N}_t(\Omega_i, z)^{\beta_i} \chi_E(z) dz = \frac{m^{\mathrm{BR}}(F_{\boldsymbol{\Omega}}^{\boldsymbol{\beta}}) w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)}.$$

1.2. **An overview of the method.** To prove Theorem 1.5. we first turn the integrals (1.9) and (1.10) into forms that fit into Mohammadi-Oh's theorem on the equidistribution of expanding horospheres (Theorem 1.6). Here in particular, for our convenience we use the HAN and NAH decompositions for G (see Section 2 for the definitions of H, A and N), which seem new to us and we name these decompositions *the generalized Iwasawa decompositions*.

Theorem 1.6 (Mohammadi-Oh, Theorem 1.7, [23]). *Suppose $\Gamma < G$ is geometrically finite. Suppose $\Gamma \backslash \Gamma N$ is closed in $\Gamma \backslash G$ and $|\mu_N^{PS}| < \infty$. For any $\Psi \in C_c^\infty(\Gamma \backslash G)$ and any $f \in C^\infty(\Gamma \backslash \Gamma N)$, we have*

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \backslash \Gamma N} \Psi(na_t) f(n) d\mu_N^{\mathrm{Leb}}(n) = \frac{m^{\mathrm{BR}}(\Psi) \mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma \backslash G)}. \tag{1.13}$$

However, Theorem 1.6 can not be directly applied, because in the statement of Theorem 1.6, the test function Ψ is assumed to be compactly supported and smooth, while in our situation, Ψ is $F_{\mathbf{\Omega},\boldsymbol{r}}$ or $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$, which are neither continuous nor compactly supported. The smoothness condition for f and Ψ is for the purpose of obtaining a version of equidistribution with exponential convergence rate. This is not needed for our purpose, as we only pursue asymptotics. By the same method from [26], the restriction for f can be relaxed to be in $L^1(\Gamma\backslash\Gamma N)$ together with some mild regularity assumption, and Ψ can be relaxed to be continuous and compactly supported; but this is still not enough for our purpose. We circumvent this technical difficulty by proving Proposition 5.2, illustrating some hierarchy structure in the space $\mathcal{W}$ of pairs of test functions (f, Ψ) where the conclusion of Theorem 1.6 holds.

Theorem 1.1 implies that certain pairs (f_0, Ψ_0) related to the counting of circles are in the space $\mathcal{W}$. An elementary geometric argument shows that $F_{\mathbf{\Omega},\boldsymbol{r}}, F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$ are dominated by Ψ_0. This together with Proposition 5.2 give us the desired Theorem 1.7, which is an extension of Theorem 1.6.

Theorem 1.7. *Let* $\Gamma < PSL(2, \mathbb{C})$ *be a discrete group with the limit set* $\Lambda(\Gamma) = \overline{\mathcal{P}}$ *and acting transitively on the circles from* $\mathcal{P}$. *Let* $\Psi = F_{\mathbf{\Omega},\mathbf{r}}$ *or* $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$, *where* $F_{\mathbf{\Omega},\mathbf{r}}$ *and* $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$ *are defined by* (1.11) *and* (1.12). *Then* $m^{\mathrm{BR}}(\Psi) < \infty$, *and*

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{\mathbb{C}} \chi_E(z)\Psi(n_z a_t)dz = \frac{m^{\mathrm{BR}}(\Psi)w(E)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}. \tag{1.14}$$

Theorem 1.5 then follows from Theorem 1.7.

Remark 2. It is desirable to prove a version of Theorem 1.6 only assuming the integrality of Ψ over the Burger-Roblin measure plus some mild restriction. While it is an exercise to relax the compactly-supported assumption to being in L^1 when the hyperbolic space has finite volume, such an extension seems much less obvious (at least to the author) if the space has infinite volume. We have made partial progress (say, Ψ can be in the Schwartz space) but haven't been able to achieve sufficient generality to encompass Theorem 1.7.

1.3. **A historical note.** Pair correlation as well as other spatial statistics have been widely used in various disciplines such as physics and biology. For instance, in microscopic physics, the Kirkwood-Buff Solution Theory [19] links the pair correlation function of gas molecules, which encodes the microscopic details of the distribution of these molecules to some macroscopic thermodynamical properties of the gas such as pressure and potential energy. In macroscopic physics, cosmologists use pair correlations to study the distribution of stars and galaxies.

Within mathematics, there is also a rich literature on the spatial statistics of point processes arising from various settings, such as Riemann zeta zeros [24], fractional parts of $\{\sqrt{n}, n \in \mathbb{Z}^+\}$ [14], directions of lattice points [9], [8], [18], [27], [21], [13], Farey sequences and their generalizations [17], [7], [5], [29], [4], [2], and translation surfaces [1], [3], [33]. Our list of interesting works here is far from inclusive. These statistics can contain rich information and yield surprising discoveries. For instance, Montgomery and Dyson's famous discovery that the pair correlation of Riemann zeta zeros agrees

with that of the eigenvalues of random Hermitian matrices, bridges analytic number theory and high energy physics.

There is a major difference between all works mentioned above and our investigation of circles here. In the above works, the underlying point sequences are uniformly distributed in their "ambient" spaces. In our case, the set of centers is fractal in nature: it is not dense in any reasonable ambient space such, as B_1, the disk centered at 0 and of radius 1. Consequently, we need different normalizations of parameters.

In some of the works above, the problems were eventually reduced to the equidistribution of expanding horospheres in finite volume hyperbolic spaces. In our case, we need an infinite version of this dynamical fact, which is Theorem 1.6, as well as to take care of certain emerging issues in the infinite volume situation. The main contribution of this paper, in the eyes of the author, is to introduce the recently rapidly developed theory of thin groups to study the fine scale structures of fractals, by displaying a thorough investigation of the well known Apollonian gaskets.

1.4. **The structure of the paper.** Section 2 gives some basic background in hyperbolic geometry. In Section 3 we set up the problem and reduce proving Theorem 1.5 to proving Theorem 1.7. In Section 4 we give a detailed discussion of some emerging conformal measures built up from the Patterson-Sullivan densities. We finish the proof of Theorem 1.7 in Section 5. Finally in Section 6 we explain how to deduce Theorem 1.3 and Theorem 1.4 from Theorem 1.7. We give complete detail for the limiting pair correlation; the limiting nearest neighbor spacing can be deduced in an analogous way and we sketch the proof.

1.5. **Notation.** We use the following standard notation. The expressions $f \ll g$ and $f = O(g)$ are synonymous, and $f \asymp g$ means $f \ll g$ and $g \ll f$. Unless otherwise specified, all the implied constants depend at most on the symmetry group Γ. The symbol $\mathbf{1}\{\cdot\}$ is the indicator function of the event $\{\cdot\}$. For a finite set $\mathcal{S}$, we denote the cardinality of $\mathcal{S}$ by $\#\mathcal{S}$.

1.6. **Ackowledgement.** Figures 3-7 were produced in a research project of Illinois Geometry Lab (IGL) [12], where Weiru Chen, Calvin Kessler and Mo Jiao were the undergraduate investigators, Amita Malik was the graduate mentor, and the author of this paper was the faculty mentor. Although we didn't use the results from [21] directly, that paper together with the data produced from the IGL project gave us the main inspiration of this paper. The technology employed in this paper is mainly from [34], [26], [23]. Thanks are also due to Prof. Curt McMullen for enlightening comments and corrections.

2. Hyperbolic 3-space and groups of isometries

We use the upper half-space model for the hyperbolic 3-space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{x + y\mathbf{i} + r\mathbf{j} : x + y\mathbf{i} \in \mathbb{C}, r \in \mathbb{R}\}.$$

The boundary $\partial\mathbb{H}^3$ of $\mathbb{H}^3$ is identified with $\mathbb{C} \cup \{\infty\}$.

The hyperbolic metric and the volume form on $\mathbb{H}^3$ are given by

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2},$$

$$dV = \frac{dxdydr}{r^3}.$$

Let $G = PSL(2, \mathbb{C})$ be the group of orientation-preserving isometries of $\mathbb{H}^3$, and let e be the identity element of G. The action of G on $\mathbb{H}^3$ is given explicitly as the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + r\mathbf{j}) = \frac{a\bar{c}|z|^2 + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d} + r^2 a\bar{c}}{|cz + d|^2 + r^2|c|^2} + \frac{r}{|cz + d|^2 + r^2|c|^2}\mathbf{j}.$$

For any two points $q_1, q_2 \in \mathbb{H}^3$, the formula for their hyperbolic distance $d(q_1, q_2)$ is

$$d(q_1, q_2) = \text{Arccosh}\left(1 + \frac{|q_1 - q_2|^2}{2\Im(q_1)\Im(q_2)}\right), \tag{2.1}$$

where $|q_1 - q_2|$ is the Euclidean distance between q_1 and q_2.

Let π_1, π_2 be the maps from G to $\mathrm{T}^1(\mathbb{H}^3), \mathbb{H}^3$ defined by

$$\pi_1(g) := g(X_1),$$
$$\pi_2(g) := g(\mathbf{j}).$$

The following subgroups of G will appear in our analysis:

(i) $A =: \left\{ a_t = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} : t \in \mathbb{R} \right\}.$

(ii) $K =: PSU(2) = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} |a|^2 + |b|^2 = 1 \right\}.$

(iii) $M =: \left\{ m_\theta = \begin{pmatrix} e^{\frac{\theta}{2}\mathbf{i}} & 0 \\ 0 & e^{-\frac{\theta}{2}\mathbf{i}} \end{pmatrix} : \theta \in [0, 2\pi) \right\}.$

(iv) $N =: \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$

(v) $H := SU(1, 1) \cup SU(1, 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where

$$SU(1, 1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} : \xi, \eta \in \mathbb{C}, |\xi|^2 - |\eta|^2 = 1 \right\}.$$

(vi) $H_0 := SU(1, 1)$, the identity component of H.

(vii) $\widetilde{A} =: \left\{ \widetilde{a}_t = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix} : t \in \mathbb{R} \right\}.$

We now explain the geometric meaning of the above groups. Let $\{X_1, X_2, X_3\}$ be an *orthonomal frame* based at $\mathbf{j}$, where X_1, X_2, X_3 are unit vectors based at $\mathbf{j}$ pointing to the negative r direction, positive y direction, and the positive x direction, respectively. Let $S \subset \mathbb{H}^3$ be the hyperbolic geodesic plane with boundary $\partial S = C(0, 1)$, where $C(z, R) \subset \mathbb{C}$ is the circle centered at z with radius R. The group G can also be identified with the orthonormal frame bundle on $\mathbb{H}^3$. The flows $\{a_t(X_1) : t \in \mathbb{R}\}, \{\widetilde{a}_t(X_3) : t \in \mathbb{R}\}$ are the geodesic flows passing X_1, X_3, respectively. The group K is the stabilizer of $\mathbf{j}$

and M is the stabilizer of X_1. The orbit $N(X_1)$ is the expanding horosphere containing X_1. The group H is the stabilizer of the geodesic plane S.

In our analysis we adopt the following decomposition for G which are particularly convenient for us:

$$G = NAH; G = HAN.$$

We call these decompositions *the generalized Iwasawa decompositions.*

We further decompose the group H via the Cartan decomposition:

$$H = M \left(\widetilde{A}^+ \cup \widetilde{A}^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) M, \tag{2.2}$$

where

$$\widetilde{A}^+ =: \left\{ \widetilde{a}_t = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} : t \in (0, \infty) \right\}.$$

For every $h \in H - M \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M$, we can write $h = m_1 a m_2$ with $m_1, m_2 \in M$ and $a \in \widetilde{A}^+ \cup \widetilde{A}^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in a unique way.

Now we show that the generalized Iwasawa decompositions parametrize G except for codimension one subvarieties. We first consider $G = NAH$. Let V be the set of all horizontal vectors and vertical vectors in $\mathrm{T}^1(\mathbb{H}^3)$, where a horizontal (vertical) vector is a vector parallel (perpendicular) to $\mathbb{C}$ in the Euclidean sense. Let $G_V = \{g \in G : g(X_1) \in V\}$. We claim the product map ρ_1:

$$N \times A \times M \times \left(\widetilde{A} \cup \widetilde{A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \times M \longrightarrow G - G_V :$$

$$\rho_1(n, a, m_1, \widetilde{A}, m_2) := nam_1 \widetilde{A} m_2 \tag{2.3}$$

is a homeomorphism.

Indeed, we notice first that the map $\pi_2 \circ \rho$ on the set

$$L^1 := \{e\} \times \{e\} \times M \times \left(\widetilde{A} \cup \widetilde{A} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \times \{e\}$$

gives an identification of L^1 with all vectors in the unit normal bundle $\mathrm{N}^1(S)$ not vertical. For any vector $u \in \mathrm{T}^1(\mathbb{H}) - V$, we can find unique elements $m_1 \in M, \widetilde{a} \in \widetilde{A}$ such that $m_1 \widetilde{a}(X_1)$ and u point to the same Euclidean direction. Next we can find a unique element $a \in A$ such that $am_1 \widetilde{a}(X_1)$ and u are based in the same horizontal plane. Then we can find a unique element $n \in N$ so that $nam_1 \widetilde{a}(X_1)$ and u are based at the same point. We observe that the actions of N, A on $\mathrm{T}^1(\mathbb{H}^3)$ preserve Euclidean directions. So in fact we have $nam_1 \widetilde{a}(X_1) = u$. The group M preserves X_1, and acts transitively and faithfully on all vectors in $T^1_e(\mathbb{H}^3)$ normal to X_1, so M can be identified with all orthonormal frames based at $\mathbf{j}$ with the first reference vector X_1. As a result, choosing a unique $m_2 \in M$ for the rightmost factor M on the left hand side of (2.3), we can take the frame $\{X_1, X_2, X_3\}$ at e to any frame at $\pi(u)$ with the first reference vector u, by the action of $nam_1 \widetilde{a} m_2$. Similarly, we have a decomposition

$G = HAN$ induced from the decomposition $G = NAH$ by the inverse map of G. This decomposition parametrizes all elements in $G - G_V^{-1}$.

3. Setup of the problem

Let $\mathcal{P} \subset \mathbb{C}$ be a bounded Apollonian gasket, and $\mathcal{C} = \mathcal{C}_{\mathcal{P}}$ be the collection of all centers from $\mathcal{P}$. Let $\mathcal{P}_t$ be the set of the circles from $\mathcal{P}$ with curvatures $< e^{-t}$ and $\mathcal{C}_t$ be the set of centers of $\mathcal{P}_t$.

Fix an open set $E \subset \mathbb{C}$ with $v(E) > 0$ and ∂E empty or piecewise smooth, and a set $\boldsymbol{\Omega} = \prod_{1 \leq i \leq k} \Omega_i \subset \mathbb{C}^k$, where $\Omega_i, 1 \leq i \leq k$ are bounded open subsets of $\mathbb{C}$ with piecewise smooth boundaries.

Let

$$\mathcal{B}_t(\Omega_i, z) := (e^{-t}\Omega_i + z) \cap \mathcal{C}_t,$$

and

$$\mathcal{N}_t(\Omega_i, z) := \#\mathcal{B}_t(\Omega_i, z).$$

We want to study

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathbf{1}\{\mathcal{N}_t(\Omega_i, z) = r_i\} \chi_E(z) dz. \tag{3.1}$$

and

$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathcal{N}_t(\Omega_i, z)^{\beta_i} \chi_E(z) dz. \tag{3.2}$$

as $t \to \infty$.

To proceed, first we choose a Kleinian group $\Gamma < PSL(2, \mathbb{C})$ whose limit set $\Lambda(\Gamma) = \mathcal{P}$, such that Γ transitively on the circles from $\mathcal{P}$. The existence of Γ can be seen as follows: let

$$\Gamma_0 = \left\langle PSL(2, \mathbb{Z}), \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix} \right\rangle.$$

One can check that the limit set of Γ_0 is the closure of the unbounded Apollonian packing $\mathcal{P}_0$, determined by three mutually tangent circles $\mathbb{R}, \mathbb{R} + \mathbf{i}, C(\mathbf{i}/2, 1/2)$, and Γ_0 acts transitively on the circles from $\mathcal{P}_0$. Since any Apollonian packing $\mathcal{P}$ can be mapped to $\mathcal{P}_0$ by a Möbius transform, the symmetry group Γ of $\mathcal{P}$ can then be taken as a conjugate of Γ_0.

Recall that S is the geodesic plane with $\partial S = C(0, 1)$, then for any isometry $g \in G$, $g(S)$ is also a geodesic plane, so in the upper half-space model, $g(S)$ is either a hemisphere or a vertical plane.

Recall the maps $\mathbf{q}$ from G to $\overline{\mathbb{H}^3}$, $\mathbf{q}_{\Re}$ from G to $\widehat{\mathbb{C}}$ defined at (1.3), (1.4). If $\infty \notin g(\partial S)$, there exists a unique geodesic l_g which traverses $g(S)$ perpendicularly. Then $\mathbf{q}(g)$ is the intersection of l_g and $g(S)$, and $\mathbf{q}_{\Re}(g)$ is the other end point of $l(g)$ besides ∞, whence we can see that the definitions for $\mathbf{q}$ and $\mathbf{q}_{\Re}$ at g with $\infty \in g(\partial S)$ are continuous extensions. So in fact both $\mathbf{q}$ and $\mathbf{q}_{\mathbb{R}}$ are continuous everywhere.

Let $\mathbf{r} = \langle r_1, \ldots, r_k \rangle, \boldsymbol{\beta} = \langle \beta_1, \ldots, \beta_k \rangle$ be multi-indices, where $r_i \in \mathbb{Z}_{\geq 0}, \beta_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq k$, and at least one component of $\mathbf{r}, \boldsymbol{\beta}$ is nonzero. Let $\Omega_i^* \subset \mathbb{H}^3$ be the "chimney"

$$\Omega_i^* := \{z + r\mathbf{j} : z \in \Omega_i, r > 1\},$$

for $1 \leq i \leq k$.

Let $\Gamma_H = \Gamma \cap H$. Since $Stab(C(0,1)) = H$ and Γ acts transitively on the circles from $\mathcal{P}$, we have
$$\mathcal{C} = \{\Re(\mathbf{q}(\gamma)) : \gamma \in \Gamma/\Gamma_H\},$$
and
$$\mathcal{C}_t = \{\Re(\mathbf{q}(\gamma)) : \gamma \in \Gamma/\Gamma_H, \Im(\mathbf{q}(\gamma)) > e^{-t}\}.$$
Therefore, we can rewrite $\mathcal{N}_t(\Omega_i, z)$ as
$$\begin{aligned}
\mathcal{N}_t(\Omega_i, z) &= \#(e^{-t}\Omega_i + z) \cap \mathcal{C}_t \\
&= \#\{\gamma \in \Gamma/\Gamma_H : \Re(\mathbf{q}(\gamma)) \in e^{-t}\Omega_i + z, \Im(\mathbf{q}(\gamma)) > e^{-t}\} \\
&= \#\{\gamma \in \Gamma/\Gamma_H : \Re(a_{-t}n_{-z}\mathbf{q}(\gamma)) \in \Omega_i, \Im(a_{-t}n_{-z}\mathbf{q}(\gamma)) > 1\} \\
&= \#\{\gamma \in \Gamma/\Gamma_H : \mathbf{q}(a_{-t}n_{-z}\gamma) \in \Omega_i^*\}. \tag{3.3}
\end{aligned}$$

Recall the definitions for the functions $F_{\mathbf{\Omega},\mathbf{r}}, F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$ on G defined by (1.11) and (1.12):
$$F_{\mathbf{\Omega},\mathbf{r}}(g) := \prod_{1 \leq i \leq k} \mathbf{1}\left\{\#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*) = r_i\right\},$$
$$F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}(g) := \prod_{1 \leq i \leq k} \#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*)^{\beta_i}.$$

Collecting (3.3),(1.11),(1.12), we have
$$\int_{\mathbb{C}} \prod_{1 \leq i \leq k} \mathbf{1}\{\mathcal{N}_t(\Omega_i, z) = r_i\}\chi_E(z)dz = \int_{\mathbb{C}} F_{\mathbf{\Omega},\mathbf{r}}(n_z a_t)\chi_E(z)dz, \tag{3.4}$$
$$\int_{\mathbb{C}} \left(\prod_{1 \leq i \leq k} \mathcal{N}_t(\Omega_i, z)^{\beta_i}\right) \chi_E(z)dz = \int_{\mathbb{C}} F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}(n_z a_t)\chi_E(z)dz. \tag{3.5}$$

At this point, we have rephrased our problem in the setting of Theorem 1.6. We restate it here:

Theorem 3.1 (Mohammadi-Oh,[23]). *Suppose $\Gamma < G$ is geometrically finite. Suppose $\Gamma\backslash\Gamma N$ is closed in $\Gamma\backslash G$ and $|\mu_N^{PS}| < \infty$. For any $\Psi \in C_c^{\infty}(\Gamma\backslash G)$ and any $f \in C^{\infty}(\Gamma\backslash\Gamma N)$, we have*
$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} \Psi(na_t)f(n)d\mu_N^{\text{Leb}}(n) = \frac{m^{\text{BR}}(\Psi)\mu_N^{PS}(f)}{m^{\text{BMS}}(\Gamma\backslash G)}. \tag{3.6}$$

Here $m^{\text{BR}}, \mu_N^{PS}, m^{\text{BMS}}$ are certain conformal measures for which we are going into detail in the next two sections. In our situation, Γ is the symmetry group of the Apollonian gasket $\mathcal{P}$, f is the characteristic funtion χ_E, and Ψ is $F_{\mathbf{\Omega},\mathbf{r}}$ or $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$. We have $\Gamma\backslash\Gamma N = N$ as $\Gamma \cap N = \{e\}$. Since Γ is geometrically finite, we have $0 < m^{\text{BMS}}(\Gamma\backslash G) < \infty$. We will also see that $\mu_N^{PS}(\chi_E) < \infty$. The issue for us to apply Theorem 1.6 is, none of the functions $f, F_{\mathbf{\Omega},\mathbf{r}}$ or $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$ is continuous. Moreover, $F_{\mathbf{\Omega},\mathbf{r}}$, $F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}$ are not compactly supported, so apriori $m^{\text{BR}}(F_{\mathbf{\Omega},\mathbf{r}})$, $m^{\text{BR}}(F_{\mathbf{\Omega}}^{\boldsymbol{\beta}})$ can be ∞. The purpose of the next two

sections is to prove Theorem 1.7, which is an extended version of Theorem 1.6. Along the way we will see that $m^{\mathrm{BR}}(F_{\mathbf{\Omega,r}}), m^{\mathrm{BR}}(F_{\mathbf{\Omega}}^{\boldsymbol{\beta}}) < \infty$.

4. CONFORMAL MEASURES

We keep all notation from previous sections. Let $\Gamma < G$ be a discrete group with the limit set $\Lambda(\Gamma) = \overline{\mathcal{P}}$ and acting transitively on the circles from $\mathcal{P}$. A family of finite measures $\{\mu_x : x \in \mathbb{H}^3\}$ is called a Γ-*invariant conformal density* of dimension $\delta_\mu > 0$ if for any $x, y \in \mathbb{H}^3, u \in \partial\mathbb{H}^3$,

$$\gamma^*\mu_x = \mu_{\gamma x}, \quad \text{and} \quad \frac{d\mu_x(u)}{d\mu_y(u)} = e^{-\beta_u(x,y)\delta_\mu},$$

where for any Borel set $F \subset \partial\mathbb{H}^n$, $\gamma^*\mu_x(F) = \mu_x(\gamma^{-1}F)$. The function β_u is the Busemann function defined as:

$$\beta_u(x,y) = \lim_{t\to\infty} d(u_t, x) - d(u_t, y),$$

where u_t is any geodesic ray tending to u as $t \to \infty$.

Two particularly important densities are the Lebesgue density $\{m_x : x \in \mathbb{H}^3\}$ and the Patterson-Sullivan density $\{\nu_x : x \in \mathbb{H}^3\}$. The Lebesgue density is a G-invariant density of dimension 2, and for each x, m_x is $Stab(x)$-invariant. The Patterson-Sullivan density $\{\nu_x\}$ is supported on the limit set $\overline{\mathcal{P}}$, and of dimension δ [31]. Both densities are unique up to scaling. We normalize these densities so that $|\nu_{\mathbf{j}}| = 1$ and $|m_{\mathbf{j}}| = \pi$.

Write $z = x + y\mathbf{i}$. We have an explicit formula for $m_{\mathbf{j}}$ in the $\mathbb{C}$ coordinate:

$$dm_{\mathbf{j}}(z) = \frac{dxdy}{(1 + x^2 + y^2)^2}. \tag{4.1}$$

Therefore, $dm_{\mathbf{j}}(z) \approx dxdy$ near 0.

The formula for $\nu_{\mathbf{j}}$ is explicitly given as the weak limit as $s \to \delta^+$ of the family of measures

$$\nu_{\mathbf{j},s} := \frac{1}{\sum_{\gamma\in\Gamma} e^{-sd(\mathbf{j},\gamma\mathbf{j})}} \sum_{\gamma\in\Gamma} e^{-sd(\mathbf{j},\gamma\mathbf{j})}\delta_{\gamma\mathbf{j}},$$

where $\delta_{\gamma\mathbf{j}}$ is the Dirac delta measure supported at the point $\gamma\mathbf{j}$.

We have the following estimate for $\nu_{\mathbf{j}}(B(z,r))$, where $B(z,r) \subset \mathbb{C}$ is the Euclidean ball centered at z with radius r (see Sec. 7 of [32]):

$$\nu_{\mathbf{j}}(B(z,r)) \ll \min\{r^\delta, 1\}. \tag{4.2}$$

By a simple packing argument, (4.2) implies $\nu_{\mathbf{j}}(l) = 0$ for any differentiable curve $l \subset \mathbb{C}$.

We also need to work with certain measures related to the conformal densities $\{m_x : x \in \mathbb{H}^3\}$ and $\{\nu_x : x \in \mathbb{H}^3\}$. For any $u \in \mathrm{T}^1(\mathbb{H}^3)$, let $u^-, u^+ \in \widehat{\mathbb{C}}$ be the starting and ending points of u. We can identify N with $\partial\mathbb{H}^3 - \{0\}$ via the map $g \to g(X_0)^+$. Let $H_1 = H/M$, then H_1 can be identified with $\partial\mathbb{H}^3 - \partial S$ via the map $g \to g(X_0)^-$. We

define measures $\mu_N^{\mathrm{PS}}, \mu_{H_1}^{\mathrm{PS}}$ as:

$$d\mu_N^{\mathrm{Leb}}(n) := e^{2\beta_{n(X_1)^+}(\mathbf{j}, n(\mathbf{j}))} dm_{\mathbf{j}}(n(X_1)^+), \tag{4.3}$$

$$d\mu_N^{\mathrm{PS}}(n) := e^{\delta\beta_{n(X_1)^+}(\mathbf{j}, n(\mathbf{j}))} d\nu_{\mathbf{j}}(n(X_1)^+), \tag{4.4}$$

$$d\mu_{H_1}^{\mathrm{PS}}(h_1) := e^{\delta\beta_{h_1(X_1)^-}(\mathbf{j}, h_1(\mathbf{j}))} d\nu_{\mathbf{j}}(h_1(X_1)^-). \tag{4.5}$$

Later on it will follow from Lemma 4.12 that $\mu_N^{\mathrm{Leb}}(n_z) = dz$, so μ_N^{Leb} is in fact a Haar measure on N.

We can lift the measure $\mu_{H_1}^{\mathrm{PS}}$ to a unique right M-invariant measure μ_H^{PS} on H satisfying: for any $f \in C_c(H_1)$, define $\hat{f} \in C_c(H)$ as

$$\hat{f}(h) = f(hM).$$

Then

$$\int_H \hat{f}(h) d\mu_H^{\mathrm{PS}}(h) = \int_{H_1} f(h_1) d\mu_{H_1}^{\mathrm{PS}}(h_1).$$

We can view H as a circle bundle over H_1. It is straightforward from the definition of μ_H^{PS} that

$$\mu_H^{\mathrm{PS}} = d\mu_{H_1}^{\mathrm{PS}} \cdot dm_M^{\mathrm{Haar}},$$

where m_M^{Haar} is the Haar measure of M with $|m_M^{\mathrm{Haar}}| = 1$.

4.1. **Finiteness of $\mu_N^{\mathrm{PS}}(N), \mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1)$ and $\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)$.** In this section we are going to show $0 < \mu_N^{\mathrm{PS}}(N), \mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1), \mu_H^{\mathrm{PS}}(\Gamma_H \backslash H) < \infty$. The > 0 part is trivial and we focus on the $< \infty$ part. We begin with a calculation:

Lemma 4.1. *For any $q \in \mathbb{H}^3$, we have $\beta_\infty(\mathbf{j}, q) = \log \Im(q)$.*

Proof. By the definition of the Buseman function,

$$\beta_\infty(\mathbf{j}, q) = \lim_{t \to \infty} d(e^{-t}\mathbf{j}, \mathbf{j}) - d(e^{-t}\mathbf{j}, q) = t - \lim_{t \to \infty} d(\mathbf{j}, e^{-t}q). \tag{4.6}$$

From the hyperbolic Disance formula (2.1),

$$\begin{aligned}
d(\mathbf{j}, e^{-t}q) &= \mathrm{Arccosh}\left(1 + \frac{e^t|\mathbf{j} - e^{-t}q|^2}{2\Im(q)}\right) \\
&= \mathrm{Arccosh}\left(1 + \frac{e^{t-\log\Im(q)}}{2}(1 + O_q(e^{-t}))\right) \\
&= t - \log\Im(q) + O_q(e^{-t}).
\end{aligned} \tag{4.7}$$

Applying (4.7) to (4.6), we obtain

$$\beta_\infty(\mathbf{j}, q) = \lim_{t \to \infty} t - (t - \log\Im(q) + O_q(e^{-t})) = \log\Im(q).$$

$\square$

Returning to (4.4), we have

$$e^{\delta\beta_{n(X_1)^+}(\mathbf{j}, n(\mathbf{j}))} = e^{\delta\beta_{n(0)}(\mathbf{j}, n(\mathbf{j}))} = e^{-\delta\beta_0(\mathbf{j}, n^{-1}(\mathbf{j}))} = e^{-\delta\beta_\infty\left(\mathbf{j}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} n^{-1}(\mathbf{j})\right)} = (|n^{-1}(0)|^2 + 1)^\delta \tag{4.8}$$

Since $\Lambda(\Gamma) = \overline{\mathcal{P}}$ is compact, the term $(|n^{-1}(0)|^2 + 1)^\delta$ is bounded on the support of $\Lambda(\Gamma) = \overline{\mathcal{P}}$ of μ_N^{PS}. As $|v_{\mathbf{j}}|$ is finite, we have $\mu_N^{\mathrm{PS}}(N) < \infty$.

Both $\Gamma_H \backslash H$ and $\Gamma \backslash G$ have one cusp, whose ranks in both Γ_H and Γ are equal to 1. Therefore $\mu_{H_1}^{\mathrm{PS}}$ is compactly supported in $\Gamma_H \backslash H_1$ from Theorem 6.3 [26]. Thus the term $e^{\delta\beta_{h_1(X_1)} - (\mathbf{j}, h_1(\mathbf{j}))}$ from (4.5) is bounded and we have $\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H) = \mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1) < \infty$.

4.2. Quasi-product Conformal Measures on $\mathrm{T}^1(\mathbb{H}^3)$.

Following Roblin [28], given two conformal measures $\{\mu_x\}, \{\mu_x'\}$, we can define a quasi-product measure $\widetilde{m}^{\mu,\mu'}$ on $\mathrm{T}^1(\mathbb{H}^3)$ by

$$d\widetilde{m}^{\mu,\mu'}(u) = e^{\delta_\mu \beta_{u^+}(o, \pi(u))} e^{\delta_{\mu'} \beta_{u^-}(o, \pi(u))} d\mu_o(u^+) d\mu_o'(u^-) ds,$$

where o is any point in $\mathbb{H}^3$, $u^-, u^+ \in \partial\mathbb{H}^3$ are the starting and ending points of the geodesic ray containing u, and $s = \beta_{u^-}(o, \pi(u))$. It is an exercise to check that

  i) The definition of $\widetilde{m}^{\mu,\mu'}$ is independent of the chosen base point o.
  ii) The measure $\widetilde{m}^{\mu,\mu'}$ is left Γ-invariant.

We can lift the measure $m^{\mu,\mu'}$ to a unique right M-invariant measure on G satisfying: for any $f \in C_c(\mathrm{T}^1(\mathbb{H}^3))$, define $\hat{f} \in C_c(G)$ as

$$\hat{f}(g) = f(g(X_1)).$$

Then

$$\int_G \hat{f}(g) dm^{\mu,\mu'}(g) = \int_{T^1(\mathbb{H}^3)} f(u) d\widetilde{m}^{\mu,\mu'}(u).$$

We can view G as a circle bundle over $\mathrm{T}^1(\mathbb{H}^3)$, and the right action of M on G preserves fibers. From the right M-invariance of $m^{\mu,\mu'}$, we have

$$dm^{\mu,\mu'} = d\widetilde{m}^{\mu,\mu'} \cdot dm_M^{\mathrm{Haar}}.$$

By the Γ-invariance, the measures $\widetilde{m}^{\mu,\mu'}, m^{\mu,\mu'}$ naturally descend to measures on $\Gamma \backslash \mathrm{T}^1(\mathbb{H}^3), \Gamma \backslash G$, for which we keep the same notation. For a Γ-invariant function F, we denote the integral $\int_{\Gamma \backslash G} F(g) dm^{\mu,\mu'}(g)$ by $m^{\mu,\mu'}(F)$.

We choose the base point $o = \mathbf{j}$. The following two quasi-product measures will appear in our analysis:

  (1) $\mu = m_{\mathbf{j}}, \mu' = \nu_{\mathbf{j}}$; we denote the measure on $\mathrm{T}^1(\mathbb{H}^3)$ by $\widetilde{m}^{\mathrm{BR}}$, and the measure on G by m^{BR}. These measures are called the Burger-Roblin measures.
  (2) $\mu = \nu_{\mathbf{j}}, \mu' = \nu_{\mathbf{j}}$; we denote the measure on $\mathrm{T}^1(\mathbb{H}^3)$ by $\widetilde{m}^{\mathrm{BMS}}$, and the measure on $\Gamma \backslash G$ by m^{BMS}. These measures are called the Bowen-Margulis-Sullivan measures.

We point out a few useful properties of these quasi-product measures.

The Burger-Roblin measures and the Bowen-Margulis-Sullivan measures are locally finite, regular Borel measures, and vanish on a countable union of submanifolds of $\mathrm{T}^1(\mathbb{H}^3)$ or G of codimension ≥ 1 (for instance, algebraic subvarieties of codimension

≥ 1). This is because locally, the Burger-Roblin measures and the Bowen-Margulis-Sullivan measures are products of measures $(m_{\mathbf{j}}, \nu_{\mathbf{j}}, m_{\mathbb{R}}^{\mathrm{Harr}}, m_M^{\mathrm{Haar}})$, each of which is locally finite, regular and vanishes on submanifolds of codimension 1 of its corresponding measure space.

Finally, we have $0 < m^{\mathrm{BMS}}(\Gamma \backslash G) < \infty$, which follows from the geometrically finiteness of Γ (see Page 270 of [32]).

4.3. Computation of m^{BR} in the generalized Iwasawa Coordinates.

The purpose of this section is to compute m^{BR} in the HAN coordinates (Proposition 4.3). We further write H into its Cartan decomposition (2.2). This decomposition provides an explicit fibration of H over H_1, with the first two factors $M \times \left(\widetilde{A}^+ \cup \widetilde{A}^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$ of (2.2) parametrize $H_1 = H/M$ except for two points $M, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M$. For this reason and for simplicity we abuse notation, writing

$$H_1 = M \times \left(\widetilde{A}^+ \cup \widetilde{A}^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right),$$

ignoring the two points M and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M$.

We first observe that the product map ρ_2:

$$L_2 := M \times \left(\widetilde{A}^+ \cup \widetilde{A}^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \times A \times N \to \mathrm{T}^1(\mathbb{H}^3),$$
$$\rho_2(m, \tilde{a}, a, n) := m\tilde{a}an(X_1)$$

embeds L_2 into an Zariski-open subset of $\mathrm{T}_1(\mathbb{H}^3)$, by a consideration similar to an earlier one for NAH decomposition, below (2.3). Under this coordinate of $\mathrm{T}^1(\mathbb{H}^3)$, we can compute

$$
\begin{aligned}
d\widetilde{m}^{BR}(h_1 a_{t_1} n_z(X_1)) =& e^{2\beta_{h_1 a_{t_1} n_z X_1^+}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})} \cdot e^{\delta\beta_{h_1 a_{t_1} n_z X_1^-}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})} \\
& dm_{\mathbf{j}}(h_1 a_{t_1} n_z X_1^+) d\nu_{\mathbf{j}}(h_1 a_{t_1} n_z X_1^-) dt \\
=& e^{2\beta_{h_1 a_{t_1} n_z 0}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})} \cdot e^{\delta\beta_{h_1 \infty}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})} \\
& dm_{\mathbf{j}}(h_1 a_{t_1} n_z 0) d\nu_{\mathbf{j}}(h_1 \infty) dt,
\end{aligned}
\tag{4.9}
$$

where $t = \beta_{h_1 \infty}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})$.

Applying Lemma 4.1 to $t = \beta_{h_1 \infty}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j})$, we obtain

$$
\begin{aligned}
t = \beta_{h_1 \infty}(\mathbf{j}, h_1 a_{t_1} n_z \mathbf{j}) =& \beta_\infty(h_1^{-1}\mathbf{j}, a_{t_1} n_z \mathbf{j}) = \beta_\infty(h_1^{-1}\mathbf{j}, \mathbf{j}) + \beta_\infty(\mathbf{j}, a_{t_1} n_z \mathbf{j}) \\
=& \log(h_1^{-1}\mathbf{j}) - t_1.
\end{aligned}
\tag{4.10}
$$

Combining (4.9) and (4.10), we obtain

$$d\widetilde{m}^{BR}(h_1 a_{t_1} n_z(X_1)) = dm_{\mathbf{j}}(h_1 a_{t_1} n_z 0) d\nu_{\mathbf{j}}(h_1 \infty) dt_1 \tag{4.11}$$

Lemma 4.2. *For any $g \in G$, consider the measure λ_g on N given by*

$$d\lambda_g(n_z) = e^{2\beta_{gn_z0}(\mathbf{j}, gn_z\mathbf{j})}dm_{\mathbf{j}}(gn_z0). \qquad (4.12)$$

Then $\lambda_g = \lambda_e$ and λ_e is a Haar measure on N.

Proof. By the G-invariance of $\{m_x\}$,

$$dm_{\mathbf{j}}(gn_z0) = dm_{g^{-1}\mathbf{j}}(n_z0) = e^{2\beta_{n_z0}(\mathbf{j}, g^{-1}\mathbf{j})}dm_{\mathbf{j}}(n_z0)$$

$$= e^{2\beta_{n_z0}(\mathbf{j}, g^{-1}\mathbf{j})}dm_{\mathbf{j}}(n_z0). \qquad (4.13)$$

Therefore,

$$d\lambda_g(n_z) = e^{2\beta_{n_z0}(g^{-1}\mathbf{j}, n_z\mathbf{j})} \cdot e^{2\beta_{n_z0}(\mathbf{j}, g^{-1}\mathbf{j})}dm_{\mathbf{j}}(n_z0) = d\lambda_e(n_z). \qquad (4.14)$$

Combining (4.12),(4.14) and (4.1), we have $d\lambda_e(n_z) = d\lambda_{n_z}(e) = d\lambda_e(e) = dz$, so λ_e is a Haar measure on N. □

Recall the definition (4.5) for $\mu_{H_1}^{\mathrm{PS}}$. We can use Lemma 4.1 to compute

$$d\mu_{H_1}^{\mathrm{PS}}(h_1) = \frac{1}{\Im(h_1^{-1}\mathbf{j})^{\delta}}d\nu_{\mathbf{j}}(h_1\infty) \qquad (4.15)$$

Collecting (4.11), Lemma 4.2 and (4.15), we obtain

$$d\widetilde{m}^{\mathrm{BR}}(h_1a_{t_1}n_zX_1) = e^{-\delta t_1}d\mu_{H_1}^{\mathrm{PS}}(h_1)dzdt_1.$$

Therefore, in the H_1ANM decomposition for G, for any $h_1 \in H_1, a_{t_1} \in A, n_z \in N, m \in M$, we have

$$dm^{\mathrm{BR}}(h_1a_{t_1}n_zm) = e^{-\delta t_1}d\mu_{H_1}^{\mathrm{PS}}(h_1)dzdt_1dm_M^{\mathrm{Haar}}(m),$$

by the right M-invariance of m^{BR}.

The decompositions H_1ANM and H_1MAN are related as follows: If $h_1a_{t_1}n_zm = h_1'm'a_{t_1'}n_{z'}$, then $h_1' = h_1, t_1' = t_1, m' = m, z' = m^{-1}z$. Therefore, in the H_1MAN decomposition, let $h_1 \in H_1, m \in M, a_t \in A, n_z \in N$, then the Burger-Roblin measure m^{BR} is given by

$$dm^{\mathrm{BR}}(h_1ma_tn_z) = e^{-\delta t}d\mu_{H_1}^{\mathrm{PS}}(h_1)dzdt_1dm.$$

Write $h = h_1m$. Since

$$d\mu_H^{\mathrm{PS}}(h_1m) = d\mu_{H_1}^{\mathrm{PS}}(h_1) \cdot dm,$$

we obtain

Proposition 4.3. *In the HAN decomposition, let $h \in H, a_t \in A, n_z \in N$. Then the Burger-Roblin measure m^{BR} is given by*

$$dm^{BR}(ha_tn_z) = e^{-\delta t}d\mu_H^{\mathrm{PS}}(h)dzdt.$$

5. Equidisribution of Expanding Horospheres

The purpose of this section is to prove Theorem 1.7, which is an extension of Theorem 1.6.

Let $\mathcal{W}$ be the set of pairs (f, Ψ) satisfying:

$$\mathcal{W} := \left\{ (f, \Psi) : \begin{array}{l} f \in L^1(\Gamma\backslash\Gamma N, \mu_N^{\mathrm{PS}}), \Psi \in L^1(\Gamma\backslash G, m^{\mathrm{BR}}), \\ \lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} \Psi(na_t)f(n)d\mu_N^{\mathrm{Leb}}(n) = \frac{m^{\mathrm{BR}}(\Psi)\mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)} \end{array} \right\}. \tag{5.1}$$

We first observe that $\mathcal{W}$ inherit some linear structure:

(i) If $(f, \Psi) \in \mathcal{W}$, then for any $\alpha_1, \alpha_2 \in \mathbb{C}$, $(\alpha_1 f, \alpha_2 \Psi) \in \mathcal{W}$.
(ii) If $(f_1, \Psi), (f_2, \Psi) \in \mathcal{W}$, then $(f_1 + f_2, \Psi) \in \mathcal{W}$.
(iii) If $(f, \Psi_1), (f, \Psi_2) \in \mathcal{W}$, then $(f, \Psi_1 + \Psi_2) \in \mathcal{W}$.

The smoothness assumption in Theorem 1.6 is for obtaining an effective convergence rate. This is not needed for our purpose here.

By the same method from [25], one can extend Theorem 1.6 to $\Psi \in C_c(\Gamma\backslash G)$ and $f \in L^1(\Gamma\backslash\Gamma N, \mu_N^{\mathrm{PS}})$ with $\lim_{\epsilon\to 0} \mu_N^{\mathrm{PS}}(f_{\epsilon+} - f_{\epsilon-}) = 0$, where

$$f_{\epsilon+}(n_z) := \sup_{|w-z|<\epsilon} f(n_w), \tag{5.2}$$

$$f_{\epsilon-}(n_z) := \inf_{|w-z|<\epsilon} f(n_w). \tag{5.3}$$

However, this is still not enough for our purpose. We need to extend Theorem 1.6 to cover some nonnegative f and Ψ, with $\Psi \in L^1(\Gamma\backslash G, m^{\mathrm{BR}})$ and non-compactly supported. Indeed, in the lattice case, the measure m^{BMS} on $\Gamma\backslash G$ is just the Haar measure, and Shah obtained Theorem 1.6 for any $f \in L^1(\Gamma\backslash\Gamma N, m_N^{\mathrm{Haar}})$ and $\Psi \in L^1(\Gamma\backslash G, m_G^{\mathrm{Haar}})$ [30]. However, it seems that removing the compactly supported assumption for Ψ is a much more delicate issue in the infinite co-volume situation. In fact, the compactly supported assumption is crucially used in proving the equidisribution theorems of expanding horospheres in the infinite co-volume case, from the works [20], [26], [23]. To see one subtlety here, compared to the lattice case, in the statement of Theorem 1.6, we have an extra factor $e^{(2-\delta)t}$, which goes to infinity as t does. We haven't been able to fully Theorem 1.6 to cover $\Psi \in L^1(\Gamma\backslash G, m^{BR})$. We circumvent this difficulty by observing some hierarchy structure in the set $\mathcal{W}$ (Proposition 5.2), which is enough for our purpose.

In Section 5.1 we prove the membership of certain pairs in $\mathcal{W}$ using Theorem 1.1, in Section 5.2 we prove some hierarchy structure in $\mathcal{W}$, and in Section 5.3 we finish the proof of Theorem 1.7.

5.1. Membership of certain pairs in $\mathcal{W}$.

Let $E \subset \mathbb{C}$ be an open set with ∂E empty or piecewise smooth, and let $\Omega \subset \mathbb{C}$ be a bounded open set with $\partial\Omega$ piecewise smooth. First we claim that $(f_0, \Psi_0) \in \mathcal{W}$, where

$$f_0(n_z) := \chi_E(z), \tag{5.4}$$

$$\Psi_0(g) := \sum_{\gamma \in \Gamma/\Gamma_H} \mathbf{1}\{\mathbf{q}(g^{-1}\gamma) \in \Omega^*\}, \tag{5.5}$$

recalling that Ω^* is the infinite chimney based at Ω (see (1.5)). We will see shortly that the pair (f_0, Ψ_0) is related to counting circles in E.

We first calculate the right hand side of (3.6) plugging in $f = f_0$ and $\Psi = \Psi_0$.

Write $g = h_1 m a_t n_z$ in the $H_1 MAN$ coordinate. From Proposition 4.3,

$$
\begin{aligned}
m^{\mathrm{BR}}(\Psi_0) &= \int_{g \in \Gamma \backslash G} \sum_{\gamma \in \Gamma_H \backslash \Gamma} \mathbf{1}\{\mathbf{q}((\gamma g)^{-1}) \in \Omega^*\} dm^{\mathrm{BR}}(g) \\
&= \int_{g \in \Gamma_H \backslash G} \mathbf{1}\{\mathbf{q}(g^{-1}) \in \Omega^*\} dm^{\mathrm{BR}}(g) \\
&= \int_{h \in \Gamma_H \backslash H} \int_{t \in \mathbb{R}} \int_{z \in \mathbb{C}} \mathbf{1}\{\mathbf{q}((h a_t n_z)^{-1}) \in \Omega^*\} \cdot e^{-\delta t} dz dt d\mu_H^{\mathrm{PS}}(h) \\
&= \int_{h \in \Gamma_H \backslash H} \int_{t > 0} \int_{z \in -\Omega} e^{-\delta t} dz dt d\mu_H^{\mathrm{PS}}(h) \\
&= \frac{1}{\delta} \cdot \mathcal{A}rea(\Omega) \mu_H^{\mathrm{PS}}(\Gamma_H \backslash H).
\end{aligned}
\tag{5.6}
$$

Next, we have $\mu_N^{\mathrm{PS}}(f_0) = w(E)$, recalling that the measure w on $\mathbb{C}$ is the pull back measure of μ_N^{PS} under the map $z \to n_z$. We also have $m^{BMS}(\Gamma \backslash G) = \widetilde{m}^{BMS}(\Gamma \backslash \mathbb{H}^3)$ and $\mu_H^{\mathrm{PS}}(\Gamma \backslash H) = \mu_{H_1}^{\mathrm{PS}}(\Gamma \backslash H_1)$. Therefore,

$$
\frac{m^{\mathrm{BR}}(\Psi_0) \mu_N^{\mathrm{PS}}(f_0)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} = \frac{\mathcal{A}rea(\Omega) \mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1) w(E)}{\delta \cdot \widetilde{m}^{\mathrm{BMS}}(\Gamma \backslash \mathbb{H}^3)}.
\tag{5.7}
$$

We now turn to the left hand side of (3.6). Recall that $\Gamma \backslash \Gamma N = N$ as $\Gamma \cap N = \{e\}$. We have

$$
\begin{aligned}
& e^{(2-\delta)t} \int_N f_0(n_z) \Psi_0(n_z a_t) dz \\
=& e^{(2-\delta)t} \int_N \chi_E(z) \sum_{\gamma \in \Gamma / \Gamma_H} \mathbf{1}\{\mathbf{q}(a_{-t} n_{-z} \gamma) \in \Omega^*\} dz \\
=& e^{(2-\delta)t} \int_N \chi_E(z) \sum_{\gamma \in \Gamma / \Gamma_H} \mathbf{1}\{z_\gamma - z \in e^{-t}\Omega; r_\gamma > e^{-t}\} dz.
\end{aligned}
\tag{5.8}
$$

Let $N(E, t) := \#\{p \in \mathcal{C}_t : \Re(p) \in E\}$, and denote the diameter of Ω by $\mathcal{D}(\Omega)$. For any $\epsilon > 0$, we let

$$
\begin{aligned}
E_{\epsilon+} &:= \{x \in \mathbb{C} : d(x, E) < \epsilon\}, \\
E_{\epsilon-} &:= \{x \in E : d(x, \partial E) > \epsilon\}.
\end{aligned}
$$

We have

$$
e^{-\delta t} \mathcal{A}rea(\Omega) \cdot N(E_{e^{-t}\mathcal{D}(\Omega)-}, t) \leq (5.8) \leq e^{-\delta t} \mathcal{A}rea(\Omega) \cdot N(E_{e^{-t}\mathcal{D}(\Omega)+}, t)
\tag{5.9}
$$

The quantity $N(*, t)$ can be estimated via the following more detailed version of Theorem 1.1:

Theorem 5.1 (Oh-Shah, Theorem 1.6, [25]). *Let $\mathcal{P}$ be a bounded Apollonian circle packing. Let $E \subset \mathbb{C}$ be an open set with no boundary or piecewise smooth boundary. Then*

$$\lim_{t \to \infty} \frac{N(E,t)}{e^{\delta t}} = \frac{\mu_{H_1}^{PS}(\Gamma_H \backslash H_1) w(E)}{\delta \cdot m^{\mathrm{BMS}}(\Gamma \backslash \mathbb{H}^3)}.$$

So comparing Theorem 1.1 and Theorem 5.1, we can see $v(E)$ and $w(E)$ are off by a constant factor:

$$v(E) = \frac{\mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1)}{\delta \cdot m^{\mathrm{BMS}}(\Gamma \backslash \mathbb{H}^3)} w(E).$$

Applying Theorem 5.1 to (5.9) with E replaced by $E_{\epsilon\pm}$, we have

$$\lim_{t \to \infty} \frac{N(E_{\epsilon\pm},t)}{e^{\delta t}} = \frac{\mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1) w(E_{\epsilon\pm})}{\delta \cdot m^{\mathrm{BMS}}(\Gamma \backslash \mathbb{H}^3)}. \tag{5.10}$$

Noting that $\lim_{\epsilon \to 0} w(E_{\epsilon+}) - w(E_{\epsilon-}) = 0$ as ∂E is piecewise smooth, and letting t goes to infinity for (5.9), we obtain

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\Gamma \backslash \Gamma N} f_0(n_z) \Psi_0(n_z a_t) dz = \frac{\mathcal{A}rea(\Omega) \mu_{H_1}^{\mathrm{PS}}(\Gamma_H \backslash H_1) w(E)}{\delta \cdot m^{\mathrm{BMS}}(\Gamma \backslash \mathbb{H}^3)}, \tag{5.11}$$

which agrees with (5.7).

5.2. **The hierarchy structure in $\mathcal{W}$.** For any $\Psi \in L^1(\Gamma \backslash G)$, let $\mathcal{S}upp(\Psi)$ be the support of Ψ and $\mathcal{D}isc(\Psi)$ be the set of discontinuities of Ψ. We aim to prove the following proposition.

Proposition 5.2. *Suppose $f \in L^1(\Gamma \backslash \Gamma N, \mu_N^{PS})$, nonnegative, and $\lim_{\epsilon \to 0} \mu_N^{PS}(f_{\epsilon+} - f_{\epsilon-}) = 0$. Suppose $\Phi \in L^1(\Gamma \backslash G, m^{\mathrm{BR}})$, nonnegative, $\|\Psi\|_{L^\infty} < \infty$, and $m^{\mathrm{BR}} \left(\overline{\mathcal{D}isc(\Psi)} \right) = 0$. If $(f, \Psi) \in \mathcal{W}$, then for any Borel measurable function $\widetilde{\Psi}$ with $0 \leq \widetilde{\Psi} \leq \Psi$ and $m^{\mathrm{BR}} \left(\overline{\mathcal{D}isc(\widetilde{\Psi})} \right) = 0$, we have $(f, \widetilde{\Psi}) \in \mathcal{W}$.*

Proof. First we prove the following claim.
Claim: for an $\epsilon > 0$, there exits $\Psi_\epsilon \in C_c(\Gamma \backslash G, m^{\mathrm{BR}})$ such that $0 \leq \Psi_\epsilon \leq \Psi$, $m^{\mathrm{BR}}(\Psi - \Psi_\epsilon) < \epsilon$ and Ψ_ϵ is supported away from the discontinuities of Ψ and $\widetilde{\Psi}$.

Since $\Gamma \backslash G$ is second countable and m^{BR} is a regular Borel measure on $\Gamma \backslash G$, we can find a compact set $K_\epsilon \subset \Gamma \backslash G$ such that

$$\int_{\Gamma \backslash G - K_\epsilon} \Psi(g) dm^{\mathrm{BR}}(g) < \epsilon/2.$$

We also choose a relatively compact open set $V_\epsilon \subset \Gamma \backslash G$ such that $K_\epsilon \subset V_\epsilon$.

Since m^{BR} is a regular Borel measure on $\Gamma \backslash G$ and

$$m^{\mathrm{BR}} \left(\overline{\mathcal{D}isc(\Psi)} \cup \overline{\mathcal{D}isc(\widetilde{\Psi})} \right) = 0,$$

we can find two open sets $U_\epsilon, U'_\epsilon \subset \Gamma \backslash G$ such that

$$\overline{\mathcal{D}isc(\Psi)} \cup \overline{\mathcal{D}isc(\widetilde{\Psi})} \subset U_\epsilon \subset \overline{U_\epsilon} \subset U'_\epsilon$$

and
$$m^{\mathrm{BR}}(U'_\epsilon) < \frac{\epsilon}{2\max\{1, \|\Psi\|_\infty\}}.$$
From the Tietze Extension Theorem, there exists a function $\Phi_\epsilon \subset C(\Gamma\backslash G)$ such that $0 \le \Psi \le 1$, $\Phi_1 \equiv 1$ on $K_\epsilon - U'_\epsilon$ and $\Phi \equiv 0$ on $\overline{U_\epsilon} \cup (\Gamma\backslash G - V_\epsilon)$.

Now set $\Psi_\epsilon = \Psi \cdot \Phi_\epsilon$, then we can see that Ψ_ϵ is compactly supported as Φ_ϵ is, Φ_ϵ is continuous because $\mathcal{S}upp(\Phi) \cap \mathcal{D}isc(\Psi) = \emptyset$, and $0 \le \Psi_\epsilon \le \Psi$. Therefore,

$$\int_{\Gamma\backslash G} \Psi(g) - \Psi_\epsilon(g) dm^{\mathrm{BR}}(g) \le \int_{\Gamma\backslash G - K_\epsilon} \Psi(g) - \Psi_\epsilon(g) dm^{\mathrm{BR}}(g) + \int_{U'_\epsilon} \Psi(g) - \Psi_\epsilon(g) dm^{\mathrm{BR}}(g)$$
$$< \epsilon/2 + \epsilon/2 = \epsilon, \tag{5.12}$$

finishing the proof of the claim.

Next, according to the comment around (5.2), for each ϵ, $(f, \Psi_\epsilon) \in \mathcal{W}$. Therefore, $(f, \Psi - \Psi_\epsilon) \in \mathcal{W}$, so that

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} f(n)(\Psi - \Psi_\epsilon)(n_z a_t) d\mu_N^{\mathrm{Leb}}(n) \le \frac{\epsilon \cdot \mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}$$

Define $\widetilde{\Psi}_\epsilon(g) := \min\{\Psi_\epsilon(g), \widetilde{\Psi}(g)\}$. We have $\widetilde{\Psi}_\epsilon \in C_c(\Gamma\backslash G)$, so that $(f, \widetilde{\Psi}_\epsilon) \in \mathcal{W}$, or

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} f(n)(\widetilde{\Psi}_\epsilon)(n_z a_t) d\mu_N^{\mathrm{Leb}}(n) = \frac{m^{\mathrm{BR}}(\widetilde{\Psi}_\epsilon) \mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}. \tag{5.13}$$

We also have

$$\int_{\Gamma\backslash G} (\widetilde{\Psi}(g) - \widetilde{\Psi}_\epsilon(g)) dm^{\mathrm{BR}}(g) \le \int_{\Gamma\backslash G} (\Psi(g) - \Psi_\epsilon(g)) dm^{\mathrm{BR}}(g) < \epsilon, \tag{5.14}$$

and

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} f(n)(\widetilde{\Psi} - \widetilde{\Psi}_\epsilon)(n_z a_t) d\mu_N^{\mathrm{Leb}}(n)$$
$$\le \lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} f(n)(\Psi - \Psi_\epsilon)(n_z a_t) d\mu_N^{\mathrm{Leb}}(n)$$
$$\le \frac{\epsilon \cdot \mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}. \tag{5.15}$$

Combining (5.14), (5.15) and (5.13), and letting $\epsilon \to 0$, we obtain

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{\Gamma\backslash\Gamma N} f(n)\widetilde{\Psi}(n_z a_t) d\mu_N^{\mathrm{Leb}}(n) = \frac{m^{\mathrm{BR}}(\widetilde{\Psi}) \mu_N^{\mathrm{PS}}(f)}{m^{\mathrm{BMS}}(\Gamma\backslash G)}, \tag{5.16}$$

so that $(f, \widetilde{\Psi}) \in \mathcal{W}$.

$\square$

5.3. **Finishing the proof of Theorem 1.7.** We begin with an elementary geometric observation, which implies that any pair of points from $\mathcal{C}_t$ can not get too close.

Observation : For any two non-intersecting hemispheres based on $\mathbb{C}$, the Euclidean distance of their apices q_1, q_2 satisfies

$$|\Re q_1 - \Re q_2| \geq \Im q_1 + \Im q_2. \tag{5.17}$$

And from the hyperbolic distance formula,

$$d(q_1, q_2) = \operatorname{Arccosh}\left(1 + \frac{|q_1 - q_2|^2}{2\Im q_1 \Im q_2}\right) \geq \operatorname{Arccosh}\left(1 + \frac{(\Im q_1 + \Im q_2)^2}{2\Im q_1 \Im q_2}\right) \geq \operatorname{Arccosh}(3). \tag{5.18}$$

From the observation (5.17), if $\mathbf{q}(g^{-1}\gamma_1), \mathbf{q}(g^{-1}\gamma_2) \in \Omega_i^*$ for $\gamma_1 \neq \gamma_2 \in \Gamma/\Gamma_H$, then $|\Re(q_1) - \Re(q_2)| \geq 2$. For each $\gamma \in \Gamma/\Gamma_H$ with $\Im(\mathbf{q}(g^{-1}\gamma)) > 1$, place a circle of radius 1 centered at $\mathbf{q}_\Re(g^{-1}\gamma)$, then these circles are disjoint. By an elementary packing argument, we have

$$\#\mathbf{q}(g^{-1}\Gamma) \cap \Omega_i^* < \frac{\pi(\mathcal{D}(\Omega_i) + 1)^2}{\pi} = (\mathcal{D}(\Omega_i) + 1)^2. \tag{5.19}$$

The functions we are interested in are $f = \chi_E$ and $\Psi = F_{\mathbf{\Omega},\mathbf{r}}, F_{\mathbf{\Omega},\boldsymbol{\beta}}$.

Suppose r_j is a nonzero component of $\mathbf{r}$, then

$$F_{\mathbf{\Omega},\mathbf{r}} = \prod_{1 \leq i \leq k} \mathbf{1}\{\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^* = r_i\} \leq \#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_j^*), \tag{5.20}$$

and

$$F_{\mathbf{\Omega},\boldsymbol{\beta}}(g) = \prod_{1 \leq i \leq k} \#(\mathbf{q}(g^{-1}\Gamma/\Gamma_H) \cap \Omega_i^*)^{\beta_i} = \sum_{\mathbf{r} > 0} \mathbf{r}^{\boldsymbol{\beta}} F_{\mathbf{\Omega},\mathbf{r}}, \tag{5.21}$$

where $\mathbf{r}^{\boldsymbol{\beta}}(g) = \prod_{1 \leq i \leq k} r_i^{\beta_i}$, and $\mathbf{r} > 0$ means all components of $\mathbf{r}$ are nonnegative, and at least one component of $\mathbf{r}$ is positive.

We notice that the right hand side of (5.20) is of the form Ψ_0 (see (5.5)), and the rightmost sum in (5.21) is a finite sum because of (5.19). So both $F_{\mathbf{\Omega},\mathbf{r}}$ and $F_{\mathbf{\Omega},\boldsymbol{\beta}}$ are dominated by (a finite linear combination of) Ψ_0. Therefore, we can apply Proposition 5.2 to $f = \chi_E, \Psi = F_{\mathbf{\Omega},\mathbf{r}}, F_{\mathbf{\Omega},\boldsymbol{\beta}}$, once we have verified that $m^{\mathrm{BR}}(\overline{\mathcal{D}isc(F_{\mathbf{\Omega},\mathbf{r}})})$, $m^{\mathrm{BR}}(\overline{\mathcal{D}isc(F_{\mathbf{\Omega},\boldsymbol{\beta}})}) = 0$. It is enough to show $m^{\mathrm{BR}}(\overline{\mathcal{D}isc(F_{\mathbf{\Omega},\mathbf{r}})}) = 0$.

Let $\mathcal{M}_{\Omega_i} := \{g \in G : \mathbf{q}(g^{-1}) \in \partial\Omega_i^*\}$. Using the NAH decomposition, we can see that $\mathcal{M}_{\Omega_i}$ is a closed submanifold of G of codimension 1, thus $m^{\mathrm{BR}}(\mathcal{M}_{\Omega_i}) = 0$.

Next, we show that

Lemma 5.3. *The immersion $\mathcal{M}_{\Omega_i} \to \pi_1(\mathcal{M}_{\Omega_i})$ is proper: for each $g \in \mathcal{M}_{\Omega_i}$, there does not exist infinitely many $\gamma_j \in \Gamma_H \backslash \Gamma$, $g_i \in \mathcal{M}$, $1 < j < \infty$, such that $\lim_{i \to \infty} \gamma_j g_i = g$.*

Proof. We argue by contradiction. Suppose there exist infinitely many $\gamma_j \in \Gamma_H \backslash \Gamma$, $g_j \in \mathcal{M}$, $1 \leq j < \infty$, such that $\lim_{j \to \infty} \gamma_j g_i = g$. Since $\mathbf{q}_\Re$ is continuous, we have $\lim_{j \to \infty} \mathbf{q}_\Re(g_j^{-1}\gamma_j^{-1}) = \mathbf{q}_\Re(g^{-1})$. We note that $\mathbf{q}(g_j^{-1}\gamma_j^{-1})$ are apices from disjoint hemispheres. Let $L := \left\{z + r\mathbf{j} \in \mathbb{H}^3 : z \in C(\mathbf{q}_\Re(g^{-1}), 1), r \in \left(\frac{\Im(\mathbf{q}(g^{-1}))}{2}, \infty\right)\right\}$. Then

$\mathbf{q}(g^{-1}) \in L$. But (5.17) implies that there can at most exist $\frac{(1+\Im(\mathbf{q}(g^{-1})))^2}{\Im(\mathbf{q}(g^{-1})^2}$ many points in L. Thus we have a contradiction. $\square$

Lemma 5.3 implies that $\overline{\pi_1(\mathcal{M}_{\Omega_i})} = \pi_1(\mathcal{M}_{\Omega_i})$, so that $m^{\mathrm{BR}}(\overline{\pi_1(\mathcal{M}_{\Omega_i})}) = 0$. Let $\mathcal{M}_{\mathbf{\Omega}} := \cup_{i=1}^k \mathcal{M}_{\Omega_i}$. As a finite union of $\mathcal{M}_{\Omega_i}$, $\mathcal{M}_{\mathbf{\Omega}}$ is closed in G and the immersion $\mathcal{M}_{\mathbf{\Omega}} \to \Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}$ is proper, so that $\Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}$ is closed in $\Gamma\backslash G$ and $m^{\mathrm{BR}}(\Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}) = 0$.

Our next lemma shows that $F_{\mathbf{\Omega},\mathbf{r}}$ is continuous outside $\Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}$, and as a corollary, $m^{\mathrm{BR}}(\mathcal{D}ist(F_{\mathbf{\Omega},\mathbf{r}})), m^{\mathrm{BR}}(\mathcal{D}ist(F_{\mathbf{\Omega},\boldsymbol{\beta}})) \leq m^{\mathrm{BR}}(\Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}) = 0$, whence we can obtain Theorem 1.7 by applying Proposition 5.2 with $f = \chi_E$, $\Psi = \Psi_0$, $\widetilde{\Psi} = F_{\mathbf{\Omega},\mathbf{r}}, F_{\mathbf{\Omega},\boldsymbol{\beta}}$.

Lemma 5.4. *Let* $\mathcal{M}_{\mathbf{\Omega}} = \cup_{i=1}^k \mathcal{M}_{\Omega_i}$, *then the function* $F_{\mathbf{\Omega},\mathbf{r}}$ *is continuous in* $\Gamma\backslash G - \Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}$.

Proof. Since the immersion $\mathcal{M}_{\mathbf{\Omega}} \to \Gamma\backslash\Gamma\mathcal{M}_{\mathbf{\Omega}}$ is proper, for any $g \in G - \Gamma\mathcal{M}_{\mathbf{\Omega}}$, there exists a simply connected open neighborhood $O_g \subset G$ of g such that $O_g \cap \Gamma\mathcal{M}_{\mathbf{\Omega}} = \emptyset$. We claim that $F_{\mathbf{\Omega},\boldsymbol{\beta}}$ is constant on $\Gamma\backslash\Gamma O_g$, by showing that for each $1 \leq i \leq k$ and each $\gamma \in \Gamma_H\backslash\Gamma$, $\mathbf{1}\{\mathbf{q}((\gamma g)^{-1}) \in \Omega_i^*\}$ is constant in O_g. Suppose $\mathbf{1}\{\mathbf{q}((\gamma g)^{-1}) \in \Omega_i^*\}$ is not constant in O_g, then there exists $g_1, g_2 \in O_g$ such that $\mathbf{q}((g_1\gamma)^{-1}) \in \Omega_i^*$ and $\mathbf{q}((g_2\gamma)^{-1}) \notin \Omega_i^*$. We observe that

$$\overline{\Omega_i^*} \cap \overline{G - \Omega_i^*} = \partial\Omega_i \cup \{\infty\}.$$

Let $p : [0, 1] \to O_g$ be a path with $p(0) = g_1$ and $p(1) = g_2$. Then for some $s \in (0, 1]$, we have $\mathbf{q}((\gamma p(s))^{-1}) \in \partial\Omega_i^* \cup \{\infty\}$. If $\mathbf{q}((\gamma p(s))^{-1}) \in \partial\Omega_i^*$, then $p(s) \in O_g \cap \Gamma\mathcal{M}_{\Omega_i}$, violating $\Omega_g \cap \Gamma\mathcal{M}_{\mathbf{\Omega}} = \emptyset$. Thus $\mathbf{q}((\gamma p(s))^{-1}) = \infty$ and we let

$$s_0 = \inf\{s \in (0, 1] : \mathbf{q}((\gamma p(s))^{-1}) = \infty\}.$$

By the continuity of p and $\mathbf{q}$, we have $\mathbf{q}((\gamma p(s_0))^{-1}) = \infty$, and by the definition of s_0, for $s < s_0$, we have $\mathbf{q}((\gamma p(s))^{-1}) \in \Omega_j^*$. Therefore, as Ω_j is bounded, $\mathbf{q}_{\Re}((\gamma p(s))^{-1})$ is bounded for $s \in (0, s_0)$. But $\mathbf{q}_{\Re}(p(s_0)) = \infty$, and this is impossible as $\mathbf{q}_{\Re}$ is a continuous map. Thus we arrived at a contradiction.

Therefore, for any γ, the function $\mathbf{1}\{\mathbf{q}((\gamma g)^{-1} \in \Omega_i^*\}$ is constant in O_g as desired. This implies $F_{\mathbf{\Omega},\mathbf{r}}$ is constant in $\Gamma\backslash\Gamma O_g$. The lemma is thus proved. $\square$

6. Pair correlation and nearest neighbor spacing

In this section we deduce Theorem 1.3 (limiting pair correlation) and Theorem 1.4 (limiting nearest neighbor spacing) from Theorem 1.7. We give full detail for the limiting pair correlation; the proof for the limiting nearest neighbor spacing is similar and we give a sketch.

6.1. **Pair correlation.** The purpose of this section is to prove Theorem 1.3. Let $E \subset \mathbb{C}$ be an open set with boundary empty or piecewise smooth, and with $v(E) > 0$. The pair correlation function $P_{E,t}(\xi)$ on the set $\mathcal{C}_t$ is defined as

$$P_{E,t}(\xi) = \frac{1}{2\#\{\mathcal{C}_t \cap E\}} \sum_{\substack{p,q \in \mathcal{C}_t \cap E \\ q \neq p}} \mathbf{1}\{|p - q| < e^{-t}\xi\}. \tag{6.1}$$

Let B_r be the disk in $\mathbb{C}$ centered at 0 with radius r. We analyze the pair correlation function $P_{E,t}$ via the following mixed 2-moment function $P_{E,t,\epsilon}$:

$$P_{E,t,\epsilon}(\xi) := \frac{e^{2t}}{2\pi\epsilon^2 \cdot \#\{\mathcal{C}_t \cap E\}} \int_{\mathbb{C}} \chi_E(z)\mathcal{N}_t(B_\epsilon, x)\mathcal{N}_t(B_\xi, x)dx - \frac{1}{2}. \tag{6.2}$$

Here ϵ is taken as a small enough positive number (say $\epsilon < \min\{\frac{1}{10}, \frac{\xi}{10}\}$), so that $\mathcal{N}_t(B_\epsilon) \leq 1$ from (5.17).

The function $P_{E,t,\epsilon}$ is an approximate to $P_{E,t}$. Indeed,

$$\int_{\mathbb{C}} \chi_E(z)\mathcal{N}_t(B_\epsilon, x)\mathcal{N}_t(B_\xi, x)dx$$

$$= \sum_{p\in\mathcal{C}_t} \int_{\mathbb{C}} \mathbf{1}\{x \in e^{-t}B_\epsilon + p\}\mathcal{N}_t(B_\xi, x)\chi_E(z)dx$$

$$\leq \sum_{p\in\mathcal{C}_t} e^{-2t}\pi\epsilon^2\mathcal{N}_t(B_{\xi+\epsilon}, p)\chi_{E_{\epsilon+}}(p)$$

$$= e^{-2t}\pi\epsilon^2 \sum_{p\in\mathcal{C}_t} \chi_{E_{\epsilon+}}(p) + e^{-2t}\pi\epsilon^2 \sum_{p\in\mathcal{C}_t} \chi_{E_{\epsilon+}}(p) \sum_{\substack{q\in\mathcal{C}_t \\ q\neq p}} \mathbf{1}\{|q - p| < e^{-t}(\xi + \epsilon)\}$$

$$\leq e^{-2t}\pi\epsilon^2\#(\mathcal{C}_t \cap E_{\epsilon+}) + e^{-2t}\pi\epsilon^2 \sum_{p\in\mathcal{C}_t\cap E_{\epsilon+}} \sum_{\substack{q\in\mathcal{C}_t \\ q\neq p}} \mathbf{1}\{|q - p| < e^{-t}(\xi + \epsilon)\}. \tag{6.3}$$

Putting (6.3) back to (6.2), we have

$$P_{E,t,\epsilon}(\xi) \leq \frac{\#(\mathcal{C}_t \cap E_{\epsilon+})}{2\#(\mathcal{C}_t \cap E)} - \frac{1}{2} + \frac{\#(\mathcal{C}_t \cap E_{\epsilon+})}{\#(\mathcal{C}_t \cap E)} P_{E_{\epsilon+},t}(\xi + \epsilon). \tag{6.4}$$

Similarly, we have

$$P_{E,t,\epsilon}(\xi) \geq \frac{\#(\mathcal{C}_t \cap E_{\epsilon-})}{2\#(\mathcal{C}_t \cap E)} - \frac{1}{2} + \frac{\#(\mathcal{C}_t \cap E_{\epsilon-})}{\#(\mathcal{C}_t \cap E)} P_{E_{\epsilon-},t}(\xi - \epsilon). \tag{6.5}$$

We can work out from (6.4) and (6.5) that

$$P_{E,t}(\xi) \leq \frac{\#(\mathcal{C}_t \cap E_{\epsilon+})}{\#(\mathcal{C}_t \cap E)} P_{E_{\epsilon+},t,\epsilon}(\xi + \epsilon) + \frac{\#(\mathcal{C}_t \cap E_{\epsilon+})}{2\#(\mathcal{C}_t \cap E)} - \frac{1}{2} \tag{6.6}$$

and

$$P_{E,t}(\xi) \geq \frac{\#(\mathcal{C}_t \cap E_{\epsilon-})}{\#(\mathcal{C}_t \cap E)} P_{E_{\epsilon-},t,\epsilon}(\xi - \epsilon) + \frac{\#(\mathcal{C}_t \cap E_{\epsilon-})}{2\#(\mathcal{C}_t \cap E)} - \frac{1}{2}. \tag{6.7}$$

Letting $t \to \infty$ and then $\epsilon \to 0$ in 6.6 and 6.7, Theorem 1.3 is proved once we have shown

$$\lim_{\epsilon\to0}\lim_{t\to\infty} P_{E_{\epsilon+},t,\epsilon}(\xi + \epsilon) = \lim_{\epsilon\to0}\lim_{t\to\infty} P_{E_{\epsilon-},t,\epsilon}(\xi - \epsilon) = P(\xi) \tag{6.8}$$

for some continuously differentiable function $P(\xi)$.

Now we analyze the limit of $P_{E,t,\epsilon}(\xi)$, as $t \to \infty$. From Theorem 5.1, we have

$$\lim_{t \to \infty} \frac{\#\mathcal{C}_t \cap E}{e^{\delta t}} = \frac{\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H) w(E)}{\delta \cdot m^{\mathrm{BMS}}(\Gamma \backslash G)}. \tag{6.9}$$

From Theorem 1.7, we have

$$\lim_{t \to \infty} e^{(2-\delta)t} \int_{\mathbb{C}} \chi_E(z) \mathcal{N}_t(B_\epsilon, x) \mathcal{N}_t(B_\xi, x) dx$$

$$= \frac{w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} \cdot \int_{\Gamma \backslash G} \left(\sum_{\gamma_1 \in \Gamma_H \backslash \Gamma} \mathbf{1}\{\mathbf{q}((\gamma g)^{-1}) \in B_\epsilon\} \right) \cdot \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}((\gamma g)^{-1}) \in B_\xi^*\} dm^{\mathrm{BR}}(g)$$

$$= \frac{w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} \cdot \int_{\Gamma_H \backslash G} \mathbf{1}\{\mathbf{q}(g^{-1}) \in B_\epsilon^\infty\} \cdot \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}((\gamma g)^{-1}) \in B_\xi^*\} dm^{\mathrm{BR}}(g). \tag{6.10}$$

Writing $g = h a_t n_z$ in the HAN decomposition, from Proposition 5.2, we have

$$(6.10) = \frac{w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} \cdot \int_{\gamma_H \backslash H} \int_{z \in B_\epsilon} \int_0^\infty e^{-\delta t} \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}(n_{-z} a_{-t} h^{-1} \gamma^{-1}) \in B_\xi^*\} dt dz d\mu_H^{\mathrm{PS}}(h). \tag{6.11}$$

Since $z \in B_\epsilon$, the condition $\mathbf{q}(n_{-z} a_{-t} h^{-1} \gamma^{-1}) \in B_\xi^*$ implies that $\mathbf{q}(a_{-t} h^{-1} \gamma^{-1}) \in B_{\xi+\epsilon}^*$. Therefore, we have

$$(6.10) \leq \frac{\pi \epsilon^2 w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} \cdot \int_{\Gamma_H \backslash H} \int_0^\infty e^{-\delta t} \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}(a_{-t} h^{-1} \gamma^{-1}) \in B_{\xi+\epsilon}^*\} dt d\mu_H^{\mathrm{PS}}(h) \tag{6.12}$$

and

$$(6.10) \geq \frac{\pi \epsilon^2 w(E)}{m^{\mathrm{BMS}}(\Gamma \backslash G)} \cdot \int_{\Gamma_H \backslash H} \int_0^\infty e^{-\delta t} \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}(a_{-t} h^{-1} \gamma^{-1}) \in B_{\xi-\epsilon}^*\} dt d\mu_H^{\mathrm{PS}}(h). \tag{6.13}$$

Define

$$P(\xi) := \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{\Gamma_H \backslash H} \int_0^\infty e^{-\delta t} \#\{\gamma \in \Gamma_H \backslash \Gamma : \mathbf{q}(a_{-t} h^{-1} \gamma^{-1}) \in B_\xi^*\} dt d\mu_H^{\mathrm{PS}}(h) - \frac{1}{2}$$

$$= \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{\Gamma_H \backslash H} \int_0^\infty e^{-\delta t} \#\{\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H) : \mathbf{q}(a_{-t} h^{-1} \gamma^{-1}) \in B_\xi^*\} dt d\mu_H^{\mathrm{PS}}(h). \tag{6.14}$$

Combining (6.9), (6.10), (6.12), (6.13), we obtain

$$P(\xi - \epsilon) \leq \lim_{t \to \infty} P_{E,t,\epsilon}(\xi) \leq P(\xi + \epsilon) \tag{6.15}$$

The definition of P is independent of the set $E \subset \mathbb{C}$, so (6.15) also holds with E replaced by $E_{\epsilon \pm}$. Thus the relation (6.8) is established once we have shown P is continuously differentiable.

First, we observe that $P(\xi)$ is indeed finite, as $\mu_H^{\mathrm{PS}}(\Gamma_H\backslash H)$ is finite and the integrand of (6.14) is bounded: for each fixed h and t, from (5.19) we have

$$\#\{\gamma \in \Gamma_H\backslash\Gamma : \mathbf{q}(a_{-t}h^{-1}\gamma^{-1}) \in B_\xi^*\} \le (2\xi + 1)^2.$$

Next, we show that the pair correlation function P is continuously differentiable.

We observe that if there exists $t > 0$ such that $\mathbf{q}(a_{-t}h^{-1}\gamma^{-1}) \in B_\xi^*$, then $\mathbf{q}(h^{-1}\gamma^{-1}) \in \mathfrak{C}_\xi \cup B_\xi^*$, where $\mathfrak{C}_\xi$ is the cone $\mathfrak{C}_\xi$ defined at (1.6).

We thus write $P(\xi)$ into two parts:

$$P(\xi) = \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H\backslash H)} \int_{\Gamma_H\backslash H} \sum_{\substack{\gamma \in \Gamma_H\backslash(\Gamma-\Gamma_H) \\ \mathbf{q}(h^{-1}\gamma^{-1})\in B_\xi^*}} \int_0^\infty e^{-\delta t} \mathbf{1}\{\mathbf{q}(a_{-t}h^{-1}\gamma^{-1}) \in B_\xi^*\} dt d\mu_H^{\mathrm{PS}}(h)$$

$$= \frac{1}{2\mu_H^{\mathrm{PS}}(\Gamma_H\backslash H)} \int_{\Gamma_H\backslash H} \sum_{\gamma \in^*\Gamma_H\backslash(\Gamma-\Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^*\} \left(1 - \left(\frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|}{\xi}\right)^\delta\right)$$

$$+ \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in \mathfrak{C}_\xi\} \left(\Im(\mathbf{q}(h^{-1}\gamma^{-1}))^\delta - \left(\frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|}{\xi}\right)^\delta\right) d\mu_H^{\mathrm{PS}}(h). \quad (6.16)$$

To proceed, we need the following lemma:

Lemma 6.1. *Define*

$$p(h,\xi) = \sum_{\gamma \in \Gamma_H\backslash(\Gamma-\Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^*\} + \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in \mathfrak{C}_\xi\}\Im(\mathbf{q}(h^{-1}\gamma^{-1}))^\delta.$$

Fixing ξ, then $p(h,\xi)$ is bounded for $h \in \Gamma_H\backslash H$.

Proof. First, from (5.19), we have

$$\sum_{\gamma_H\backslash\Gamma-\Gamma_H} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^*\} < (2\xi + 1)^2. \quad (6.17)$$

Next, let $\mathfrak{C}_\xi^{t_1,t_2}$ be the truncated cone

$$\mathfrak{C}_\xi^{t_1,t_2} := \{z + r\mathbf{j} \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{\xi}, t_1 < r \le t_2\}.$$

Recall the definition of $\mathfrak{C}_\xi$ at (1.6). An elementary exercise in hyperbolic geometry shows that the 2-neighborhood of $\mathfrak{C}_\xi$ (the set of all points in $\mathbb{H}^3$ having hyperbolic distance < 2 to $\mathfrak{C}_\xi$) is contained in the cone

$$\widetilde{\mathfrak{C}}_\xi := \left\{z + r\mathbf{j} \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{e^2\xi}, 0 < r \le e^2\right\}, \quad (6.18)$$

and the 2-neighborhood of $\mathfrak{C}_\xi^{t_1,t_2}$ is contained in the truncated cone

$$\widetilde{\mathfrak{C}}_\xi^{t_1,t_2} := \left\{z + r\mathbf{j} \in \mathbb{H}^3 : \frac{r}{|z|} > \frac{1}{e^2\xi}, \frac{t_1}{e^2} < r \le t_2e^2\right\}.$$

Therefore, for each $0 < t < 1$,

$$\sum_{\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in \mathfrak{C}_\xi\} \Im(\mathbf{q}(h^{-1}\gamma^{-1}))^\delta$$

$$= \sum_{n=0}^\infty \sum_{\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in \mathfrak{C}_\xi^{\frac{1}{2^{n+1}}, \frac{1}{2^n}}\} \Im(q(h^{-1}\gamma^{-1}))^\delta$$

$$\leq \sum_{n=0}^\infty \frac{\mathcal{V}ol(\widetilde{\mathfrak{C}}_\xi^{\frac{1}{2^{n+1}}, \frac{1}{2^n}})}{4\pi} \frac{1}{2^{n\delta}} \tag{6.19}$$

$$\ll \sum_{n=0}^\infty \frac{1}{2^{n\delta}} < \infty, \tag{6.20}$$

where in (6.19) we used a packing (by hyperbolic balls) argument combined with (5.18), and here $\mathcal{V}ol(\widetilde{\mathfrak{C}}_\xi^{\frac{1}{2^{n+1}}, \frac{1}{2^n}})$ is the hyperbolic volume of $\widetilde{\mathfrak{C}}_\xi^{\frac{1}{2^{n+1}}, \frac{1}{2^n}}$. $\qquad\square$

Now we show that $P(\xi)$ is differentiable. For small $\epsilon > 0$,

$$\frac{P(\xi + \epsilon) - P(\xi)}{\epsilon} =$$

$$\frac{1}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi\} \cdot \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta \left(\frac{1}{\xi^\delta} - \frac{1}{(\xi+\epsilon)^\delta}\right)}{\epsilon} d\mu_H^{\mathrm{PS}}(h)$$

$$+ \frac{1}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in (B_{\xi+\epsilon}^* - B_\xi^*)\} \cdot \frac{1 - \left(\frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|}{\xi+\epsilon}\right)^\delta}{\epsilon}$$

$$+ \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in (\mathfrak{C}_{\xi+\epsilon} - \mathfrak{C}_\xi)\} \cdot \frac{\Im(\mathbf{q}(h^{-1}\gamma^{-1}))^\delta - \left(\frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|}{\xi+\epsilon}\right)^\delta}{\epsilon} d\mu_H^{\mathrm{PS}}(h)$$

$$= \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi\} \cdot \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} (1 + O_\xi(\epsilon)) d\mu_H^{\mathrm{PS}}(h)$$

$$+ O_\xi \left(\int_{h \in \Gamma_H \backslash H} p(h, \xi + \epsilon) - p(h, \xi) d\mu_H^{\mathrm{PS}}(h) \right). \tag{6.21}$$

Noting that

$$\sum_{\gamma \in \gamma_H \backslash (\Gamma - \Gamma_H)} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi\} \cdot \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} \ll_\xi p(h, \xi),$$

and letting $\epsilon \to 0^+$, we have

$$\lim_{\epsilon \to 0^+} \frac{P(\xi + \epsilon) - P(\xi)}{\epsilon} = \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\substack{\gamma \in \gamma_H \backslash \Gamma - \Gamma_H \\ \mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi}} \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} d\mu_H^{\mathrm{PS}}(h),$$

$$(6.22)$$

once we have shown that the term $O(\cdot)$ from (6.21) goes to 0 as $\epsilon \to 0^+$. Indeed, since $p(h, \xi)$ is bounded with respect to h and monotone with respect to ξ, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{\epsilon \to 0^+} \int_{h \in \Gamma_H \backslash H} p(h, \xi + \epsilon) - p(h, \xi) d\mu_H^{\mathrm{PS}}(h)$$

$$= \sum_{\gamma \in \gamma_H \backslash \Gamma - \Gamma_H} \int_{h \in \Gamma_H \backslash H} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in (\partial B_\xi^* - (B_\xi + \mathbf{j}))\} d\mu_H^{\mathrm{PS}}(h)$$

$$+ \sum_{\gamma \in \gamma_H \backslash \Gamma - \Gamma_H} \int_{h \in \Gamma_H \backslash H} \mathbf{1}\{\mathbf{q}(h^{-1}\gamma^{-1}) \in (\partial \mathfrak{C}_\xi - (B_\xi + \mathbf{j}))\} \cdot \Im(\mathbf{q}(h^{-1}\gamma^{-1})^\delta d\mu_H^{\mathrm{PS}}(h).$$

$$(6.23)$$

We can check that for each $\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H)$, the set $H_{\gamma, \xi} := \{h \in H : \mathbf{q}(h^{-1}\gamma^{-1}) \in \partial(B_\xi^\infty \cup \mathfrak{C}_\xi)\}$ is contained in an algebraic subvariety of H of codimension 1. Therefore, $\mu_H^{\mathrm{PS}}(H_{\gamma, \xi}) = 0$ for each $\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H)$, so that (6.23)=0, and (6.22) is established.

By a similar consideration, we can also show

$$\lim_{\epsilon \to 0^+} \frac{P(\xi) - P(\xi - \epsilon)}{\epsilon} = \frac{\delta}{2\mu_H^{\mathrm{PS}}(\Gamma_H \backslash H)} \int_{h \in \Gamma_H \backslash H} \sum_{\substack{\gamma \in \gamma_H \backslash \Gamma - \Gamma_H \\ \mathbf{q}(h^{-1}\gamma^{-1}) \in B_\xi^* \cup \mathfrak{C}_\xi}} \frac{|\mathbf{q}_\Re(h^{-1}\gamma^{-1})|^\delta}{\xi^{\delta+1}} d\mu_H^{\mathrm{PS}}(h).$$

Theorem 1.3 is thus proved.

6.2. **Nearest Neighbor Spacing.** As usual we let $E \subset \mathbb{C}$ be an open set with no boundary or piecewise smooth boundary, and with $v(E) > 0$. For any $p \in \mathcal{C}_t$, let $d_t(p) = \min\{|p - q| : q \in \mathcal{C}_t, q \neq p\}$. The nearest neighbor spacing function $Q_{E,t}$ is defined by

$$Q_{E,t}(\xi) = \frac{1}{\#\{\mathcal{C}_t \cap E\}} \sum_{p \in \mathcal{C}_t \cap E} \mathbf{1}\{d_t(p) < e^{-t}\xi\}. \qquad (6.24)$$

We sketch our analysis for $Q_{E,t}$, which is a very similar fashion as we did for the pair correlation function. The function $Q_{E,t}(\xi)$ can be approximated by the following function

$$Q_{E,t,\epsilon}(\xi) :=$$

$$1 - \frac{e^{2t}}{\pi \epsilon^2 \cdot \#\{\mathcal{C}_t \cap E\}} \int_{\mathbb{C}} \chi_E(z) \mathbf{1}\left\{\#(a_{-t} n_z \mathbf{q}(\Gamma) \cap B_\epsilon^*) = 1\right\} \mathbf{1}\left\{\#(a_{-t} n_z \mathbf{q}(\Gamma) \cap B_\xi^*) = 0\right\} dz.$$

$$(6.25)$$

Indeed, one can check that

$$Q_{E,t}(\xi + \epsilon) \leq Q_{E,t,\epsilon}(\xi) \leq Q_{E,t}(\xi - \epsilon). \tag{6.26}$$

Applying Theorem 1.7 to $Q_{E,t,\epsilon}$ and letting $\epsilon \to \infty$, we obtain Theorem 1.4. The continuity of Q follows from that, by the Dominated Convergence Theorem,

$$\limsup_{\epsilon \to 0^{\pm}} |Q(\xi + \epsilon) - Q(\xi)| \ll_{\xi} \sum_{\gamma \in \Gamma_H \backslash (\Gamma - \Gamma_H)} \int_{\Gamma_H \backslash H} \int_0^{\infty} e^{-\delta t} \mathbf{1}\{\mathbf{q}(a_{-t}h^{-1}\gamma) \in \partial B_{\xi}^*\} dt d\mu_H^{\mathrm{PS}}(h)$$
$$= 0.$$

References

[1] J. S. Athreya and J. Chaika. The distribution of gaps for saddle connection directions. *Geom. Funct. Anal.*, 22(6):1491–1516, 2012.

[2] Jayadev Athreya, Sneha Chaubey, Amita Malik, and Alexandru Zaharescu. Geometry of Farey-Ford polygons. *New York J. Math.*, 21:637–656, 2015.

[3] Jayadev S. Athreya, Jon Chaika, and Samuel Lelièvre. The gap distribution of slopes on the golden L. In *Recent trends in ergodic theory and dynamical systems*, volume 631 of *Contemp. Math.*, pages 47–62. Amer. Math. Soc., Providence, RI, 2015.

[4] Jayadev S. Athreya, Cristian Cobeli, and Alexandru Zaharescu. Radial density in Apollonian packings. *Int. Math. Res. Not. IMRN*, (20):9991–10011, 2015.

[5] Volker Augustin, Florin P. Boca, Cristian Cobeli, and Alexandru Zaharescu. The h-spacing distribution between Farey points. *Math. Proc. Cambridge Philos. Soc.*, 131(1):23–38, 2001.

[6] Christopher J. Bishop and Peter W. Jones. Hausdorff dimension and Kleinian groups. *Acta Math.*, 179(1):1–39, 1997.

[7] Florin P. Boca, Cristian Cobeli, and Alexandru Zaharescu. A conjecture of R. R. Hall on Farey points. *J. Reine Angew. Math.*, 535:207–236, 2001.

[8] Florin P. Boca, Alexandru A. Popa, and Alexandru Zaharescu. Pair correlation of hyperbolic lattice angles. *Int. J. Number Theory*, 10(8):1955–1989, 2014.

[9] Florin P. Boca and Alexandru Zaharescu. On the correlations of directions in the Euclidean plane. *Trans. Amer. Math. Soc.*, 358(4):1797–1825, 2006.

[10] Jean Bourgain and Elena Fuchs. A proof of the positive density conjecture for integer Apollonian circle packings. *J. Amer. Math. Soc.*, 24(4):945–967, 2011.

[11] Jean Bourgain and Alex Kontorovich. On the local-global conjecture for integral Apollonian gaskets. *Invent. Math.*, 196(3):589–650, 2014. With an appendix by Péter P. Varjú.

[12] Weiru Chen, Mo Jiao, Calvin Kessler, Amita Malik, and Xin Zhang. Spatial statistics of apollonian gaskets. *Submitted. https://arxiv.org/abs/1705.06212*, 05 2017.

[13] Daniel El-Baz, Jens Marklof, and Ilya Vinogradov. The distribution of directions in an affine lattice: two-point correlations and mixed moments. *Int. Math. Res. Not. IMRN*, (5):1371–1400, 2015.

[14] Noam D. Elkies and Curtis T. McMullen. Gaps in $\sqrt{n}$ mod 1 and ergodic theory. *Duke Math. J.*, 123(1):95–139, 2004.

[15] Elena Fuchs. *Arithmetic properties of Apollonian circle packings.* ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Princeton University.

[16] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: number theory. *J. Number Theory*, 100(1):1–45, 2003.

[17] R. R. Hall. A note on Farey series. *J. London Math. Soc. (2)*, 2:139–148, 1970.

[18] Dubi Kelmer and Alex Kontorovich. On the pair correlation density for hyperbolic angles. *Duke Math. J.*, 164(3):473–509, 2015.

[19] Frank P. Kirkwood, John G.; Buff. The statistical mechanical theory of solutions. I. *The Journal of Chemical Physics*, 19(6):774–777, 6 1951.

[20] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 24(3):603–648, 2011. With an appendix by Oh and Nimish Shah.

[21] Jens Marklof and Ilya Vinogradov. Directions in hyperbolic lattices. *arXiv*, 09 2014.

[22] Curtis T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[23] Amir Mohammadi and Hee Oh. Matrix coefficients, counting and primes for orbits of geometrically finite groups. *J. Eur. Math. Soc. (JEMS)*, 17(4):837–897, 2015.

[24] H. L. Montgomery. The pair correlation of zeros of the zeta function. pages 181–193, 1973.

[25] Hee Oh and Nimish Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Invent. Math.*, 187(1):1–35, 2012.

[26] Hee Oh and Nimish A. Shah. Equidistribution and counting for orbits of geometrically finite hyperbolic groups. *J. Amer. Math. Soc.*, 26(2):511–562, 2013.

[27] Morten S. Risager and Anders Södergren. Angles in hyperbolic lattices : The pair correlation density. *arXiv*, 09 2014.

[28] Thomas Roblin. Sur l'ergodicité rationnelle et les propriétés ergodiques du flot géodésique dans les variétés hyperboliques. *Ergodic Theory Dynam. Systems*, 20(6):1785–1819, 2000.

[29] Zeev Rudnick and Xin Zhang. Gap distributions in circle packings. *Mu nster J. of Math.*, 10:131–170, 2017.

[30] Nimish A. Shah. Uniformly distributed orbits of certain flows on homogeneous spaces. *Math. Ann.*, 289(2):315–334, 1991.

[31] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[32] Dennis Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153(3-4):259–277, 1984.

[33] Caglar Uyanik and Grace Work. The distribution of gaps for saddle connections on the octagon. *Int. Math. Res. Not. IMRN*, (18):5569–5602, 2016.

[34] Dale Winter. Mixing of frame flow for rank one locally symmetric spaces and measure classification. *Israel J. Math.*, 210(1):467–507, 2015.

Department of Mathematics, University of Illinois, 1409 W Green Street, Urbana, IL 61801

E-mail address: xz87@illinois.edu