1 A dynamical outlook on orbital counting problems

This section is loosely based on the survey [Oh].
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Abstract

We expose the strategy, as explicited by Duke, Rudnick and Sarnak [DRS], of reinterpreting orbital counting problems as equidistribution problems, and give examples of some underlying equidistribution problems on the modular surface.

Let G be a non-compact connected semisimple Lie group with finite center and let $X = G/K$ be its Riemannian symmetric space, endowed with G-invariant Riemannian metric d_X. A guiding example through this section will be the hyperbolic plane $\mathbb{H} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, realized via the isomorphism $[g]\mathrm{SO}(2) \mapsto [g].i = \frac{ai+b}{ci+d}$. Let $\Gamma < G$ be a lattice, e.g. $\mathrm{SL}(2,\mathbb{Z})$ in $\mathrm{SL}(2,\mathbb{R})$.

Counting along discrete orbits in X...

Denote by $\mathcal{O}$ the discrete Γ-orbit at the base point $x_0 = [K]$, i.e. $\mathcal{O} = \Gamma x_0$. The orbital counting function $N_{\mathcal{O}} : \mathbb{R}_{>0} \to \mathbb{N}$, defined by

$$N_{\mathcal{O}}(T) \;=\; \#\left\{\gamma \in \Gamma : d_X(x_0, \gamma x_0) \leq T\right\},$$

records the (finite) number of points along the orbit $\mathcal{O}$ that are contained in the ball

$$B_T := B_T(x_0) = \{x \in X : d_X(x_0, x) \leq T\}$$

of radius T and center x_0 in X. As T grows, so does the volume of the Riemannian balls in the continuous family $\{B_T\}_{T>0}$. An orbital counting problem is to determine the asymptotic growth rate of

$$N_{\mathcal{O}}(T) \;=\; \#\left(\mathcal{O} \cap B_T\right) \quad \text{for } \; T \to \infty.$$

...via equidistribution

Consider the K-orbit $K\Gamma/\Gamma$ in G/Γ and its translates by elements $[g] \in G/K$. We say that $[g]K\Gamma/\Gamma$ equidistributes in G/Γ with respect to the Haar measure dg as $[g] \to \infty$ if, for any $\psi \in C_c(G/\Gamma)$,

$$\int_{K\Gamma/\Gamma} \psi([g]k)dk \longrightarrow \int_{G/\Gamma} \psi(g)dg \qquad \text{as} \;\; [g] \to \infty.$$

Here, dk denotes the Haar measure on G/Γ that is supported on the closed orbit $K\Gamma/\Gamma = K/(K \cap \Gamma)$. We will assume that all Haar measures are normalized, so that

$$\mathrm{vol}\,(G/\Gamma) \;=\; \mathrm{vol}\,(K/(K\cap\Gamma)) \;=\; 1.$$

Theorem 1. *If $[g]K\Gamma/\Gamma$ equidistributes in G/Γ as $[g]$ escapes to infinity, then*

$$N_{\mathcal{O}}(T) \sim \; \mathrm{vol}(B_T) \qquad \textit{as} \;\; T \to \infty.$$

The main object of this section is the proof of Theorem 1, which is exposed in the next subsection. In Subsection 3, we attempt to illustrate the dynamics underlying the equidistribution phenomenon in two examples on the hyperbolic plane. In the final part of this section, we conclude with some comments on the range of contexts where the above strategy may be applied, among others the case of Apollonian packings.

A disclaimer

In reality, the Riemannian symmetric space context we introduced is too restrictive. Indeed, the essential arguments in the proof of Theorem 1 are the existence of Haar measures on quotients of locally compact groups and a regularity assumption on the boundaries of growing compact sets B_T. More explicitly, Theorem 1 holds for

> G locally compact Hausdorff, $\Gamma < G$ discrete, $H < G$ a closed subgroup such that G and H are unimodular[1], and $\{B_T\}_{T>0}$ a continuous family of compact sets in G/H such that $\mathrm{vol}(B_T) \to \infty$ as $T \to \infty$ and such that $\{B_T\}$ is admissible in the following sense :

[1] This is in particular satisfied if $\Gamma < G$ and $\Gamma \cap H < H$ are both lattices.

Definition 2. *A continuous family $\{B_T\}_{T>0}$ of compact subsets in G/H is said to be* **well-rounded** *if, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{U}$ of identity in G such that for*

$$B_T^+ := \bigcup_{g \in \mathcal{U}} g.B_T \quad \textit{and} \quad B_T^- := \bigcap_{g \in \mathcal{U}} g.B_T,$$

$$\frac{\mathrm{vol}(B_T^+ \setminus B_T^-)}{\mathrm{vol}(B_T)} < \varepsilon$$

holds for every $T > 0$.

It is easy to verify that

$$1 - \varepsilon \ < \ \frac{\mathrm{vol}(B_T^-)}{\mathrm{vol}(B_T)} \ \leq \ \frac{\mathrm{vol}(B_T^+)}{\mathrm{vol}(B_T)} \ < \ 1 + \varepsilon \tag{1}$$

holds if and only if $\{B_T\}$ is well-rounded. Continuous families of Riemannian balls are well-rounded. As an explicit example, the reader may keep in mind that hyperbolic balls in $\mathbb{H}$ satisfy $\mathrm{vol}(B_T) \sim \pi e^T$ as $T \to \infty$.

1.1 Proof of Theorem 1

Proof. Recall that x_0 is the base point $[K] \in G/K = X$. In lieu of $N_{\mathcal{O}}$, introduce a more general counting function $F_T : G \times \mathbb{R}_{>0} \to \mathbb{N}$, defined by

$$F_T(g) \ = \sum_{\gamma \in \Gamma/(\Gamma \cap K)} 1_{B_T}(g\gamma x_0) \ = \sum_{\substack{\gamma \in \Gamma/(\Gamma \cap K) \\ \gamma x_0 \in B_T(g^{-1}x_0)}} 1,$$

that counts all the points in $\mathcal{O}$, without multiplicity, lying in translates of the ball B_T. Observe that F_T descends to a function on G/Γ and that, for any $T > 0$,

$$F_T(\Gamma) = N_{\mathcal{O}}(T).$$

Our aim is to show that the normalised counting function

$$f_T(\cdot) := \frac{F_T(\cdot)}{\mathrm{vol}(B_T)} \quad \text{on} \ \ G/\Gamma$$

verifies

$$\lim_{T \to \infty} f_T(\Gamma) = 1.$$

In a first step, we show that the equidistribution of the $K\Gamma/\Gamma$-translates yields the weak convergence

$$\lim_{T\to\infty} \langle f_T, \psi\rangle = \langle 1, \psi\rangle$$

for any $\psi \in C_c(G/\Gamma)$. In fact, by standard properties of Haar measures on quotients,

$$\begin{aligned}
\langle f_T, \psi\rangle &= \int_{G/\Gamma} \left(\sum_{\Gamma/(\Gamma\cap K)} \frac{1_{B_T}(g\gamma x_0)}{\mathrm{vol}(B_T)} \right) \psi(g)\ dg \\
&= \int_{G/(\Gamma\cap K)} \frac{1_{B_T}(g x_0)}{\mathrm{vol}(B_T)}\ \psi(g)\ dg \\
&= \int_{G/K} \int_{K/(\Gamma\cap K)} \frac{1_{B_T}(gkx_0)}{\mathrm{vol}(B_T)}\ \psi(gk)\ dk\ d[g] \\
&= \int_{G/K} \int_{K\Gamma/\Gamma} \frac{1_{B_T}([g])}{\mathrm{vol}(B_T)}\ \psi([g]k)\ dk\ d[g] \\
&= \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} \left(\int_{K\Gamma/\Gamma} \psi([g]k)\ dk \right) d[g]
\end{aligned}$$

and by letting $\mathrm{vol}(B_T) \to \infty$ (as happens when $T \to \infty$), $[g]$ can escape to infinity. Therefore, assuming equidistribution,

$$\lim_{T\to\infty} \frac{1}{\mathrm{vol}(B_T)} \int_{B_T} \left(\int_{G/\Gamma} \psi(g)\ dg \right) dx = \langle 1, \psi\rangle. \tag{2}$$

To pass from weak convergence to pointwise convergence, define a modified counting function F_T^+ for the modified data given by replacing B_T with B_T^+ (cf. Def. 2 for the notation). Let $\varepsilon > 0$ and pick a symmetric neighborhood of identity $\mathcal{U}$ (as in Def. 2) and a test function $\psi \in C_c(G/\Gamma)$ that is non-negative, supported on $\mathcal{U}\Gamma$ and normalized such that $\int \psi = \langle 1, \psi\rangle = 1$.

From the definitions, it follows that for every $g \in \mathcal{U}$,

$$F_T(\Gamma) \le F_T^+(g\Gamma)$$

and therefore

$$F_T(\Gamma) \le \langle F_T^+, \psi\rangle.$$

Then, by (1),

$$f_T(\Gamma) \;\le\; \frac{\mathrm{vol}(B_T^+)}{\mathrm{vol}(B_T)}\langle f_T^+, \psi\rangle < (1+\varepsilon)\langle f_T^+, \psi\rangle,$$

which, together with (2), implies

$$\limsup_{T\to\infty} f_T(\Gamma) \le \; (1+\varepsilon)\langle 1, \psi\rangle = 1+\varepsilon.$$

By defining similarly F_T^- for B_T^-, one obtains

$$1-\varepsilon \;\le \liminf_{T\to\infty} f_T(\Gamma) \;\le\; \limsup_{T\to\infty} f_T(\Gamma) \le \; 1+\varepsilon.$$

Since ε can be chosen arbitrarily small,

$$\lim_{T\to\infty} f_T(\Gamma) = 1, \text{ i.e. } N_{\mathcal{O}}(T) \sim \mathrm{vol}(B_T) \text{ as } T \to \infty.$$

$\square$

1.2 Some illustrative equidistribution results

In this subsection, we describe two equidistribution problems related to the homogeneous dynamics of the modular surface, and deduce the corresponding orbital counting results from Theorem 1. Set $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$,

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$N = \left\{ n_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} : s \in \mathbb{R} \right\}, \qquad K = \mathrm{Stab}_G(i) = \mathrm{SO}(2) \simeq S^1.$$

The Poincaré upper half-plane is endowed with the Riemannian metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and hyperbolic measure $d\mu = \frac{dxdy}{y^2}$. The standard fundamental domain for Γ on $\mathbb{H}$ is the hyperbolic triangle

$$\left\{ z = x+iy \in \mathbb{H} : |z| > 1,\; x \in \left[-\frac{1}{2}, \frac{1}{2}\right) \right\} \cup \left\{ z : |z| = 1,\; x \le 0 \right\}.$$

We shall also denote by μ the resulting measure on the orbifold $\Gamma\backslash\mathbb{H}$.

The group G admits a Haar measure, that can be expressed in the coordinates of the Iwasawa decomposition $G = NAK$ as $dg = e^{-t} ds dt dk$, and that projects to the hyperbolic measure on $\mathbb{H}$. We again denote by dg the resulting Haar measure on $\Gamma\backslash G$. The latter quotient may be identified with the unit tangent bundle of $\Gamma\backslash\mathbb{H}$, i.e. $\mathrm{T}^1(\Gamma\backslash\mathbb{H}) \simeq \Gamma\backslash G$.

Via this identification, we realize the geodesic flow on the unit tangent bundle as the action of A on $\Gamma\backslash G$ by right multiplication, and the horocycle flow as the action of N on $\Gamma\backslash G$ by right multiplication.

We conclude this introduction with two remarks. First, for our study orbits translated by a sequence that escapes to infinity, it suffices to take (a_t) as the escaping sequence. For instance, in G/K, this is clear because of the Cartan decomposition $G = KAK$ (where recall that K is compact). Second, for the application of Theorem 1, we note that both K and N are unimodular closed subgroups of G and that, as previously noted, hyperbolic balls are well-rounded.

Equidistribution of spheres in $\Gamma\backslash\mathbb{H}$

The sphere in $\mathbb{H}$ of radius t and center the basepoint i is the set

$$S_t := S_t(i) = \{z \in \mathbb{H} : d_{\mathbb{H}}(i, z) = t\} = \{ka_t.i : k \in K\}$$

as $d_{\mathbb{H}}(i, a_t.i) = d_{\mathbb{H}}(i, e^t i) = t$. Let μ_t denote the normalized Haar measure supported on $[S_t] \in \Gamma\backslash\mathbb{H}$. In other words, on the modular surface, for any $\psi \in C_c(\Gamma\backslash\mathbb{H})$,

$$\int_K \psi([ka_t.i])dk \;=\; \int_{[S_t]} \psi d\mu_t.$$

Theorem 3. *For any $\psi \in C_c(\Gamma\backslash\mathbb{H})$,*

$$\int_{[S_t]} \psi d\mu_t \;\longrightarrow \int_{\Gamma\backslash\mathbb{H}} \psi d\mu \qquad \textit{as} \quad t \to \infty.$$

In practice, one lifts ψ to the unit tangent bundle and shows that the convergence

$$\int_{\Gamma\backslash\Gamma K} \widetilde{\psi}(\Gamma k a_t)dk \longrightarrow \int_{\Gamma\backslash G} \widetilde{\psi} dg \quad \text{as} \quad t \to \infty$$

is a consequence of the mixing of the geodesic flow. The equidistribution, together with Theorem 1, yields the lattice point counting result

$$\#\{\gamma \in \Gamma : d_{\mathbb{H}}(i, \gamma.i) \leq T\} \sim \pi e^T \qquad \text{as} \quad T \to \infty.$$

Equidistribution of periodic horocycles in $\Gamma\backslash G$

This time, we consider the periodic orbits of the horocycle flow $\mathrm{T}^1(\Gamma\backslash\mathbb{H})$ that are the horizontal parallel lines in the fundamental domain with vectors pointing upwards. These periodic horocycles are parametrized by their Euclidean height, so that at height t, the corresponding periodic horocycle is

$$C_t = \Gamma N a_t.i$$

and has length e^{-t}. Therefore, as we go up towards the cusp, the horocycles become shorter, while as $t \to -\infty$, they become longer. One notes that as a low-lying periodic horocycle is brought back via an isometry in Γ to the fundamental domain, its trajectory becomes quite complicated, with the orbit still having length e^{-t}. The following result is, again, a consequence of Howe-Moore's Theorem.

Theorem 4. *For any $\psi \in C_c(\Gamma\backslash G)$,*

$$\int_{\Gamma\backslash\Gamma N} \psi(\Gamma n a_t) dn \;\longrightarrow \int_{\Gamma\backslash G} \psi dg \quad \textit{as} \;\; t \to -\infty.$$

Fix a base horocycle $x_0 = [N]$, and consider the discrete orbit Γx_0. In light of Theorem 1, the number of horocycles in the orbit that meet growing well-rounded compact sets B_T has asymptotic growth

$$\#(\Gamma x_0 \cap B_T) \sim \mathrm{vol}(B_T) \quad \text{as} \;\; T \to \infty.$$

In this last example, G/N is not a symmetric space, and this gives a concrete illustration of the generality of the strategy exposed in this section.

1.3 Further remarks

It has been an *ad hoc* assumption that the discrete orbits we consider come from lattices. If we consider instead thin subgroups, i.e. discrete subgroups of infinite covolume, then it is a characteristic feature of their orbits that they contain much fewer points. And in fact, the conclusion of Theorem 1 depends on Γ being a lattice. It is an easy exercise to check this with examples in Euclidean space.

A natural example of thin group arises from the symmetries of the Apollonian packing. Determining the asymptotics of the number of circles in a fixed packing $\mathcal{P}$ bounded by some curvature can be formulated as an orbital counting problem, and it has been shown in [KO] that

$$\#\{C \in \mathcal{P} : \mathrm{curv}(C) \leq T\} \sim c_{\mathcal{P}} T^{\delta},$$

where[2] $\delta \approx 1.30569$ is the Hausdorff dimension of the fractal set $\overline{\bigcup_{C\in\mathcal{P}} C}$, by relying on the strategy exposed in this note. The authors prove in [KO] the equidistribution of expanding closed horospheres in hyperbolic 3-space, with convergence not to the Haar measure of $\Gamma\backslash G$, but to a measure determined by the Patterson-Sullivan base eigenfunction, which plays the rôle of the constant function 1 as the base eigenfunction of the Laplacian in the lattice case.

Finally, we would also like to remark that the strategy described in this section also finds application in the study of integral and rational points on homogeneous varieties. This is discussed in the survey [Oh].

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