Gap Distribution of Directions in Some Schottky Groups

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The study of spatial statistics originates in mathematical physics, and has received attention also in analytic number theory and probability.

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In the Euclidean setting, the problem can be formulated as:

Question

For a fixed vector $\vec{w}$ in $\mathbb{R}^2$, consider the following increasing sequence of finite subsets of the unit circle:

$$A(N) = \left\{ \frac{\vec{v} + \vec{w}}{\mid \vec{v} + \vec{w} \mid} : \vec{v} \in \mathbb{Z}^2, \mid \vec{v} + \vec{w} \mid < N \right\}$$

What can we say about the distribution of $A(N)$, as $N \to \infty$?

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Using equidistribution of flows in some homogeneous space, Marklof and Strömbergsson (Ann. Math 2011) determined a class of spatial statistics. Among them is the gap distribution.

Let $d_1, d_2, \cdots, d_{\#A(N)}$ be the gaps from $A(N)$. Define the gap distribution function

$$F_N(s) = \frac{\#\{d_i : d_i / \frac{2\pi}{\#A(N)} < s\}}{\#A(N)}$$

Theorem (Marklof-Strömbergsson, 2011)

As $N \to \infty$, $F_N(s)$ pointwise converges to a continuous function $F(s)$. If $\vec{w} \notin \mathbb{Q}^2$, F agrees with the limiting gap distribution of $\sqrt{n}$(mod 1).

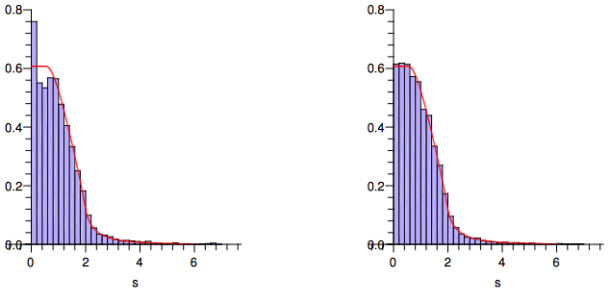

Figure: Left: The distribution of gaps in the sequence $\sqrt{n}$ mod 1, $n = 1 \cdots 7765$, vs. the Elkies-McMullen distribution. Right: Gap distribution for the directions of the vectors $(m - \sqrt{2}, n) \in \mathbb{R}^2$ with $m \in \mathbb{Z}, n \in \mathbb{Z}_{\geqslant 0}, (m - \sqrt{2})^2 + n^2 < 4900$. The continuous curve is the Elkies-McMullen distribution.

The study of spatial statistics is extended to the setting of hyperbolic lattices of finite covolume by Boca-Popa-Zaharescu, Kelmer-Kontorovich and Marklof-Vinogradov:

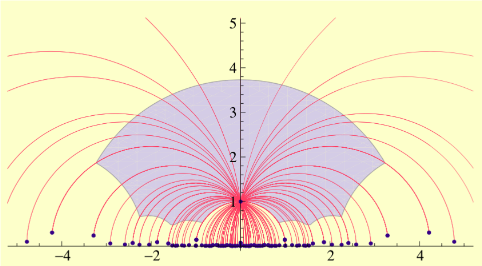

Figure: Directions of lattice points observed from $\mathbf{i}$. Picture by Kelmer-Kontorovich

Question

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We consider the following group: Let Λ be a Schottky group generated by three hyperbolic reflections, with isometric circles C_1, C_2, C_3

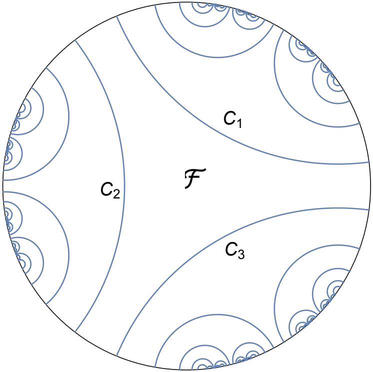
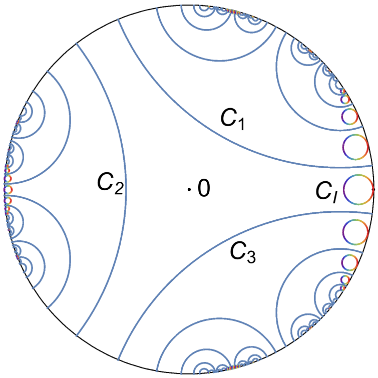


Figure: A hyperbolic reflection group



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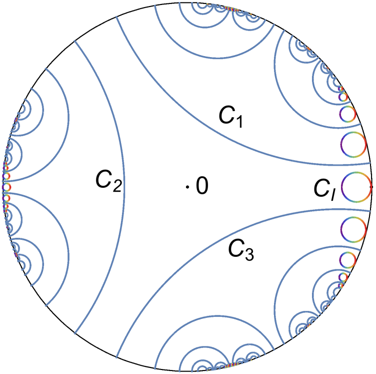

Figure: A hyperbolic reflection group

Let $A(N)$ be the collection of tangencies from circles with curvatures $(1/\text{radius}) < N$. We want to study the gap distributions of $A(N)$.

Let δ be the critical exponent of Λ, which agrees with the Hausdorff dimension of the closure of the set of all tangencies. There are $\sim cN^{2\delta}$ in total, so the average gap is $\frac{1}{cN^{2\delta}}$.

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Therefore, we need to define the gap distribution function for $A(N)$ to be

$$F_N(s) = \frac{\#\{d_i : d_i/\frac{1}{N^2} < s\}}{\#A(N)}$$

Theorem (Z)

As $N \to \infty$, $F_N(s)$ pointwise converges to $F(s)$, where F is a continuous, nonnegative function which is supported away from 0 and $\lim_{s\to\infty} F(s) = 1$.

Histograms of $\frac{dF_N}{ds}$ for various N:

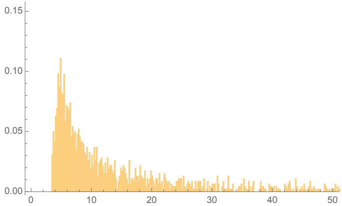

Figure: $N = 10^3$

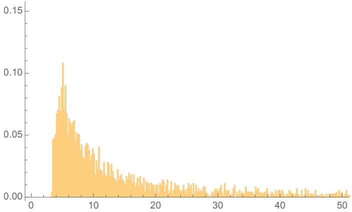

Figure: $N = \sqrt{2} \times 10^3$

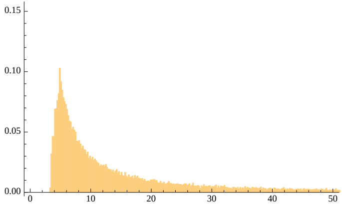

Figure: $N = 10^4$

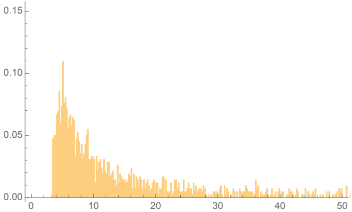

Figure: $N = \sqrt{2} \times 10^3$ and tangencies are taken from $[0.695204, 2.980334]$

Ingredients of the proof:

- Reduction to a hyperbolic lattice point counting problem in $PSU(1,1)$. A typical such problem is to count lattice points asymptotically in an expanding subset of $PSU(1,1)$

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- Tools from homogeneous dynamics (Oh-Shah's Theorem (JAMS, 2013), mixing of the geodesic flow under Bowen-Margulis-Sullivan density)

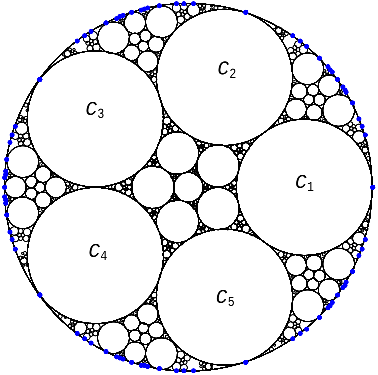

Figure: Tangencies in an Apollonian 9-circle packing

Theorem (Rudnick-Z, 2015)

There exists a limiting gap distribution for tangencies from an Apollonian 9-circle packing.

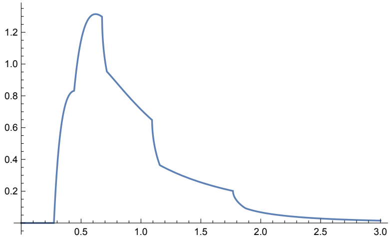

Figure: The density $F'(s)$ of the gap distribution for Apollonian 9-circle packings.

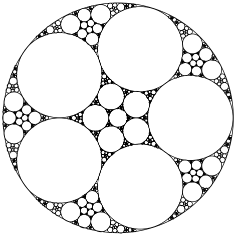

Figure: An Apollonian 9-Circle Packing

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- $\#A_{1,2}(s) = \#\{\gamma \in \Lambda : \kappa(\gamma C_1) < N, \kappa(\gamma C_2) < N, \kappa(\gamma C_3) > N, \kappa(\gamma C_4) > N, \kappa(\gamma C_5) > N, d(\gamma(\alpha_i), \gamma(\alpha_j)) < \frac{s}{N^2}\}$

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- Under the coordinate of Cartan decomposition, the above conditions can be rephrased as

$$(\phi_1(\gamma), \phi_2(\gamma), t(\gamma)) \in \Omega_s(N),$$

where

$$\Omega_s(N) = 2\log N \cdot \Omega_s(1) = \{(\phi_1, \phi_2, 2\log N \cdot t) : (\phi_1, \phi_2, t) \in \Omega_s(1)\}$$

Cartan Decomposition

Let $\mathbb{D}$ be the Poincaré disc with the metric $ds^2 = \frac{4(dx^2+dy^2)}{(1-(x^2+y^2))^2}$. The orientation-preserving symmetry group of $\mathbb{D}$ is

$$G = PSU(1,1) = \left\{ \begin{pmatrix} \xi & \eta \\ \bar{\eta} & \bar{\xi} \end{pmatrix} \middle| |\xi|^2 - |\eta|^2 = 1 \right\} \cong PSL_2(\mathbb{R}).$$

Let

$$K = \left\{ k_\phi = \begin{pmatrix} e^{\frac{\phi \mathbf{i}}{2}} & \\ & e^{\frac{-\phi \mathbf{i}}{2}} \end{pmatrix} \middle| \phi \in [0, 2\pi) \right\},$$

$$A = \left\{ a_t = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \middle| t \in [0, \infty) \right\}.$$

Recall the Cartan decompositon $G = KA^+K$ that each $g \in G$ can be written in a unique way as

$$g = k_{\phi_1(g)} a_{t(g)} k_{\pi - \phi_2(g)}$$

with $\phi_1(g), \phi_2(g) \in [0, 2\pi)$ and $t(g) > 0$. The Haar measure is given by $dg = e^t d\phi_1 d\phi_2 dt$.

Joint equidistribution of Lattices of Finite Covolume in Cartan Decomposition

Theorem (Good)

Let Λ be a lattice of $SU(1,1)$ of finite covolume. Let $\mathcal{I}, \mathcal{J}$ be intervals in $[0, 2\pi)$. As $N \to \infty$,

$$\#\{\gamma \in \Lambda : \phi_1(\gamma) \in \mathcal{I}, \phi_2(\gamma) \in \mathcal{J}, t(\gamma) < N\} \sim \frac{l(\mathcal{I})l(\mathcal{J})}{4\pi^2 V(\Lambda)} e^N,$$

where l is the standard arclength measure.

Joint equidistribution of Lattices of infinite Covolume in Cartan Decomposition

Theorem (Bourgain-Kontorovich-Sarnak, Oh-Shah, Mohammadi-Oh)

Let Λ be a lattice of $SU(1,1)$ of infinite covolume, with critical exponent δ. Let $\mathcal{I}, \mathcal{J}$ be intervals in $[0, 2\pi)$. As $N \to \infty$,

$$\#\{\gamma \in \Lambda : \phi_1(\gamma) \in \mathcal{I}, \phi_2(\gamma) \in \mathcal{J}, t(\gamma) < N\} \sim \frac{\nu(\mathcal{I})\nu(\mathcal{J})}{4\pi^2 V(\Lambda)} e^{\delta N},$$

where ν is the Patterson-Sullivan measure on $[0,2\pi)$.

Motivating problems: How are the circles from an Apollonian circle packings distributed?

Theorem (Oh-Shah, Invent. Math. 2012)

There is a finite Borel measure μ on the plane, such that for any region $\mathcal{R}$ with smooth boundary, $K_{\mathcal{R}}(N)$ the number of circles in $\mathcal{R}$ with curvature bounded by N has asymptotic growth

$$K_{\mathcal{R}}(N) \sim \mu(\mathcal{R}) N^{\delta_0}$$

where $\delta_0 \approx 1.305688$ is the Hausdorff dimension of the circle packing.

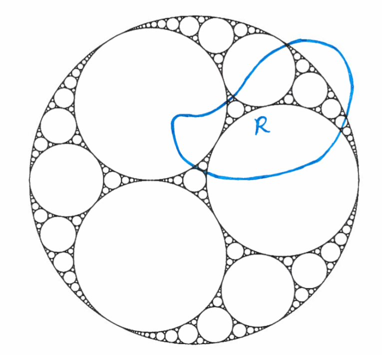

Figure: A region $\mathcal{R}$ with smooth boundary

Beyond equidistribution, what else can we say? Let X_N be the centers of circles from $\mathcal{P}$. We want to study the spatial statistics on X_N.

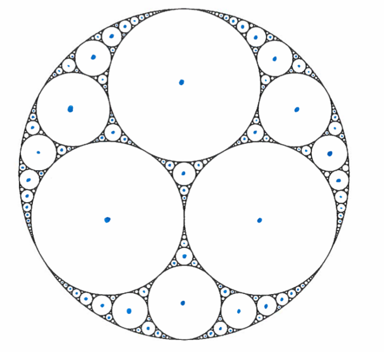

Figure: Centers

Electrostatic energy

The electrostatic energy of X_N is defined to be

$$E(X_N) = \sum_{\substack{p,q \in X_N \\ p \neq q}} \frac{1}{|p-q|}$$

The energy E depends on both the global distribution of points as well as a moderate penalty if two points are too close to each other. More generally, one can consider the Riesz s-energy:

$$E_s(X_N) = \sum_{\substack{p,q \in X_N \\ p \neq q}} \frac{1}{|p-q|^s}$$

Question

What's the behavior of $E_s(X_N)$ as $N \to \infty$? Is there an asymptotic growth?

Nearest neighbor spacing statistics

Let $d_{p,N}$ denote the distance of p to the remaining points of X_N.
A typical $d_{p,N}$ should have scale $1/N$.
We define the nearest spacing measure $\nu(X_N)$ on $[0, \infty)$ by

$$\nu(X_N) := \frac{1}{\#X_N} \sum_{p \in X_N} \delta_{d_{p,N} N}.$$

where δ_ξ is a delta mass at $\xi \in \mathbb{R}^+$.

Question

Is there a limiting distribution for $\nu(X_N)$ as $N \to \infty$?