

Self-Intersection Numbers of Closed Geodesics in Infinite Volume Hyperbolic Surfaces



Yujia Yan, Peiwen Zhang, Xianhao Zhang, Graduate Mentor: Hadrian Quan, Faculty Mentor: Xin Zhang

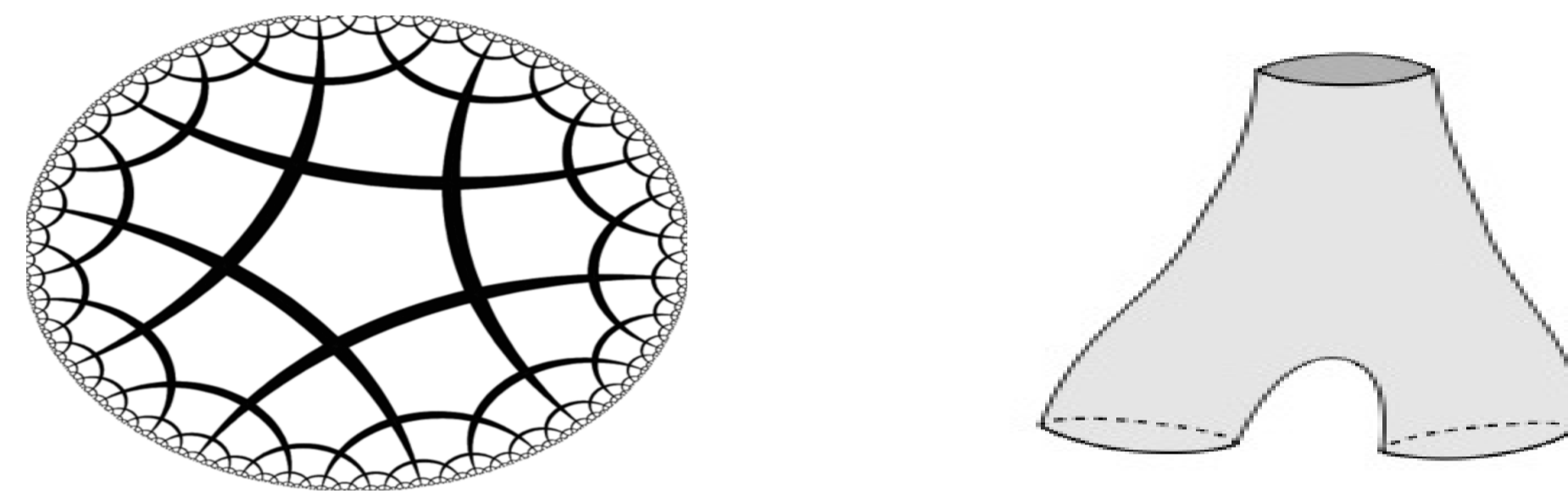
Introduction and Background

Goal

Obtain numerical estimates for the number of times a typical closed geodesic intersects itself on a pair of pants.

Pair of Pants

A pair of pants is a surface which is homeomorphic to the three-holed sphere with no boundary. The name comes from considering one of the removed disks as the waist and the two others as the cuffs of a pair of pants. Pairs of pants are used as building blocks for compact surfaces in various theories.



Background Information

- Hyperbolic geometry studies curved surfaces which locally resembles the hyperbolic disc, like the image above and left
- A geodesic is the shortest straight line that connects two points in hyperbolic surface, and its path can be represented by a word written in the given surface's alphabet.
- The number of times a geodesic intersects itself is the self-intersection number.

Theorem 1

On a surface with finite volume, non-empty boundary, and negative Euler Characteristic χ , the proportion of words ω of word length L such that

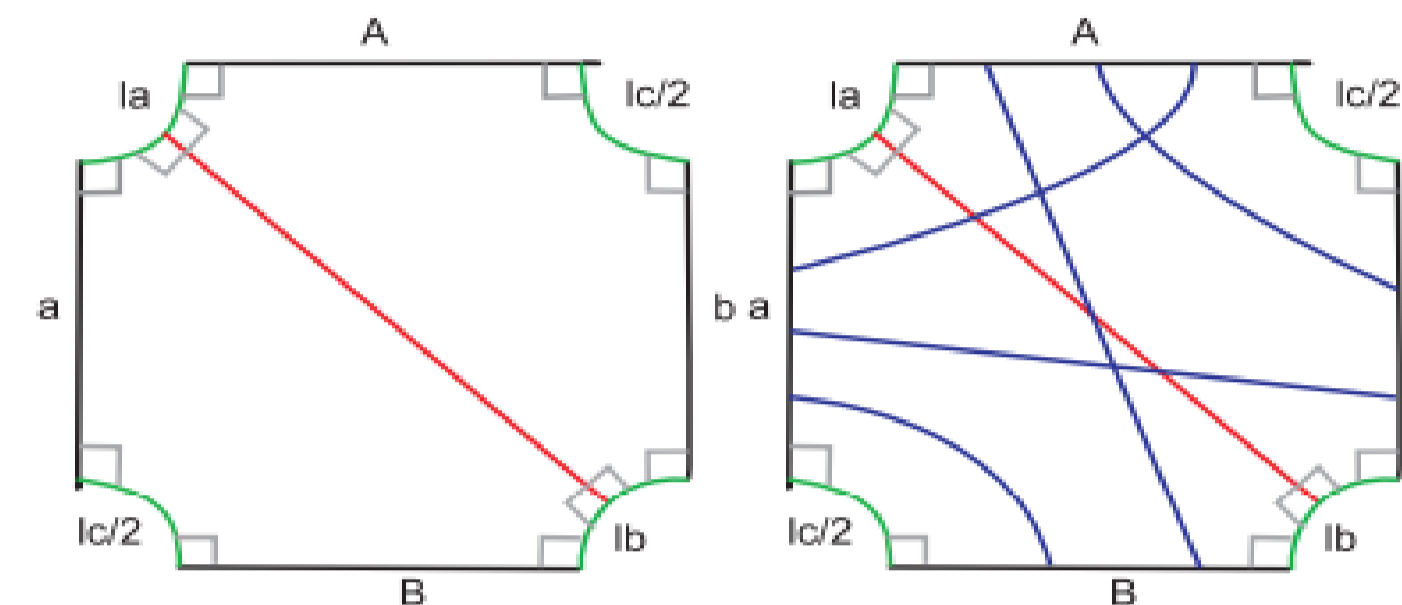
$$a < \frac{SI(\omega) - \kappa_\chi L^2}{L^{3/2}} < b \quad (*)$$

converges. In fact

$$\frac{|\{\omega : \omega \text{ satisfies } (*)\}|}{|\{\omega : WL(\omega) = L\}|} \rightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

where κ_χ depends only on the Euler characteristic of the surface. This means when L is very large, the distribution of self-intersection of all free homotopy classes of word length L is close to a Gaussian with mean $\kappa_\chi L^2$ and standard deviation $\sigma L^{3/2}$.

Methods and Results

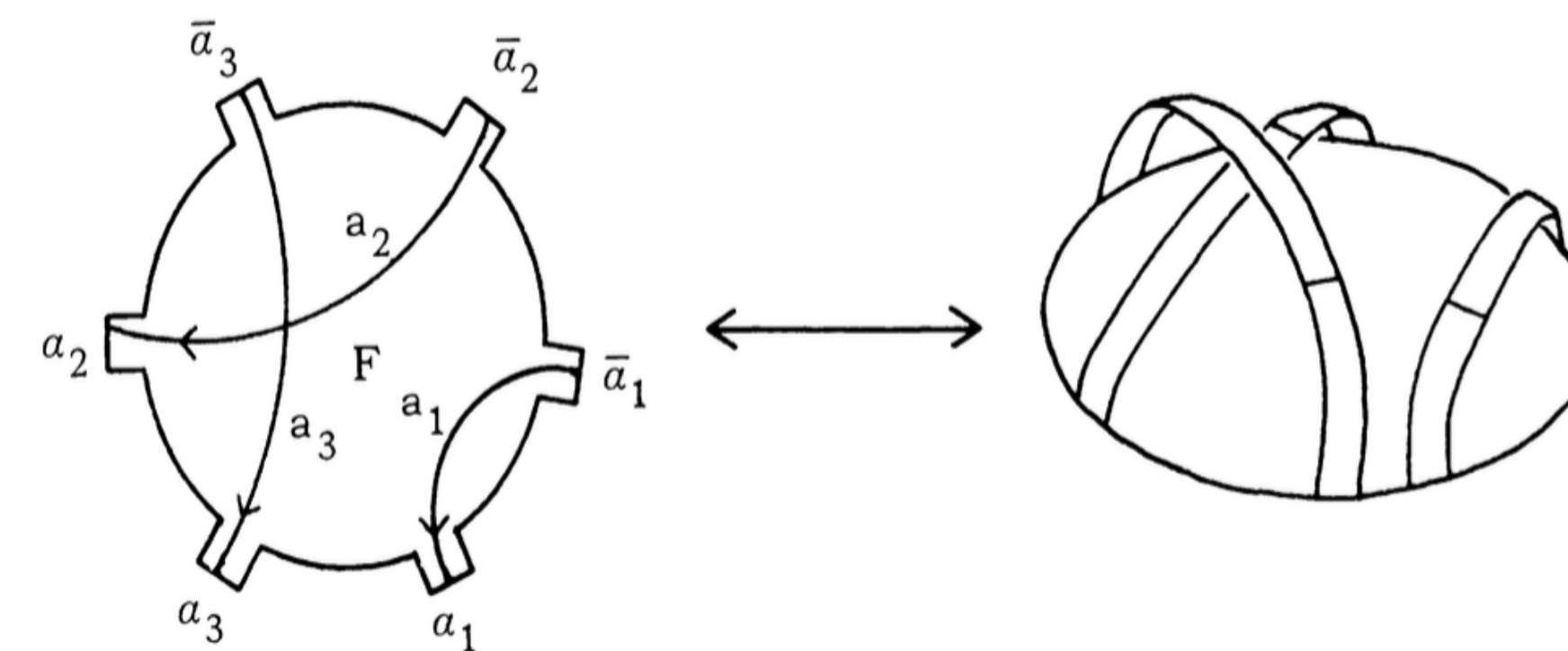


The geometric length of a geodesic on the pair of pants is completely determined by the lengths of the boundary curves. If we set $C = \min\{la, lb, lc/2\}$, then Chas proves that for every geodesic ω in the pair of pants, its word length is related to geometric length as follows

$$GL(\omega) \geq C \cdot WL(\omega)$$

Combinatorial intersections:

Following an algorithm of Cohen-Lustig we have computed the number of self-intersection for closed curves whose words have geometric length from 6 to 30. As we keep adding the volume of our sample size, the histograms tend towards a bell-shaped normal distribution which matches the theoretical results of Steven Lalley.



Theorem 2. The method we use is the algorithm comes from Cohen and Lustig, and is modified from the original algorithm of Birman-Series to extend to surfaces with non-empty boundary. Our program does the following:

INPUT:

1) Given an ordered alphabet for surface word, and word of some length expressed in this alphabet.

OUTPUT:

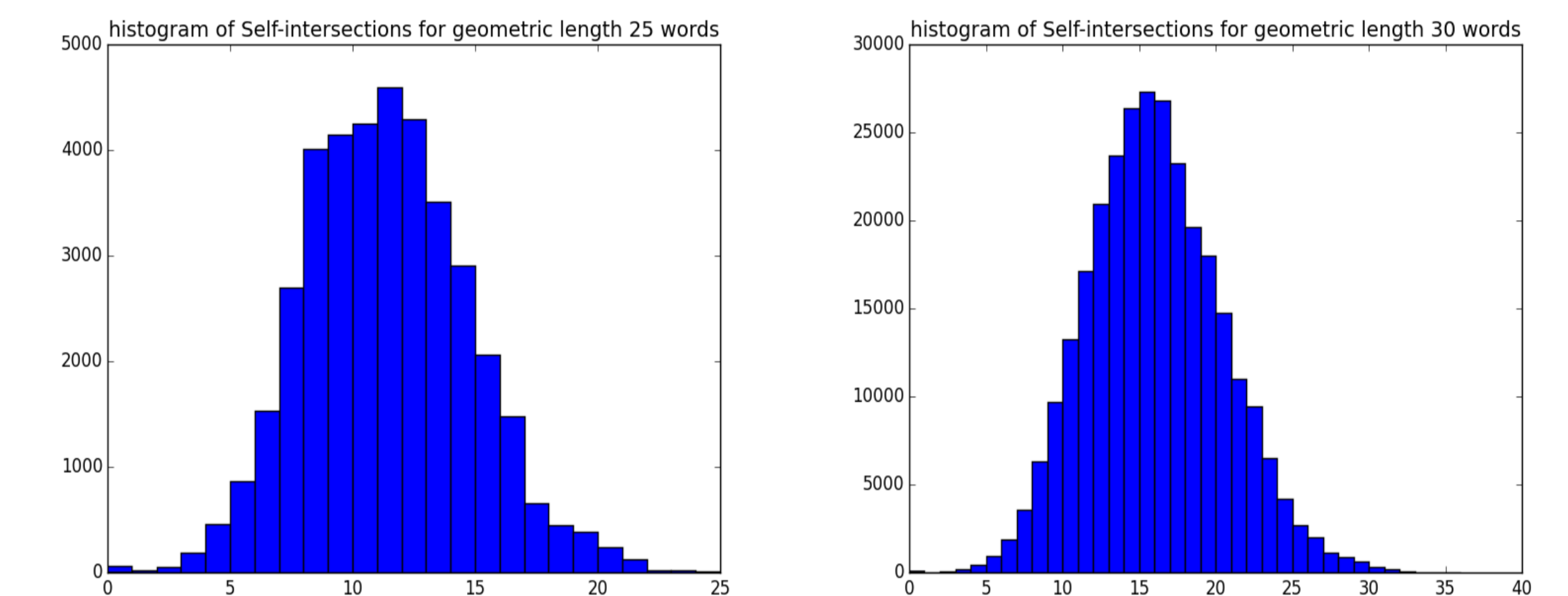
1) Generate all the possible cyclic permutations

2) Generate all possible **linking pairs** as defined by Birman-Series

3) Count the number of self-intersection points, as determined by a linked pairs of words satisfying an ordering on words

Future Directions

0.1 Conclusions



The histogram of geometric length of given length words appears to tend towards a normal distribution, as suggested by Chas' numerical results.

0.2 Future Plans

Since we have already gone through the "pair of pants" model and sampling with geometry length of the words, we are going to do the sampling on other infinite hyperbolic surfaces afterwards. By generating the self-intersection number by geometry length in the "pair of pants" model, we are going to study if there is a fixed relationship between them on infinite volume surfaces, similar to theorem 1.

Further Questions

If a result like Theorem 1 is true for the pair of pants, what is the proper scaling exponent?

Answering Question Our current results suggest the scaling is not $L^{3/2}$, but we have not yet gathered enough data to conclude what other power it should be. The value appears to be some $2 < \delta < 5/2$.

References

- [1] Chas, M. Lalley, S.P. Self-intersections in combinatorial topology: statistical structure. Invent. math. (2012) 188: 429. doi:10.1007/s00222-011-0350-7
- [2] Chas, Moira. "Relations between word length, hyperbolic length and self-intersection number of curves on surfaces.?" Recent Advances in Mathematics RMS-Lecture Notes 21 (2015): 45-75. APA
- [3] Cohen, Marshall, and Martin Lustig. "Paths of geodesics and geometric intersection numbers. I." Combinatorial group theory and topology (Alta, Utah, 1984) 111 (2016): 479-500.
- [4] Birman, Joan S., and Caroline Series. "An algorithm for simple curves on surfaces." J. London Math. Soc.(2) 29.2 (1984): 331-342.